*Journal of Statistical Physics, Vol. 115, Nos. 5/6, June 2004 (© 2004)*

# Equilibrium Fluctuations for Zero-Range-Exclusion Processes

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*Received May 2, 2003; accepted January 6, 2004*

We introduce a Markovian particle system which is a kind of lattice gas on **Z** consisting of particles carrying energy and whose dynamics is a combination of those of an exclusion process (for particles) and a zero-range process (for energy). It has two conserved quantities, the number of particles and the total energy. The process is reversible relative to certain product probability measures, but of non-gradient type. It is proved that under hydrodynamic scaling the equilibrium fluctuation fields of two conserved quantities converge in law to an infinite dimensional Ornstein–Uhlenbeck process.

**KEY WORDS:** Fluctuation fields; hydrodynamic scaling; non-gradient system; diffusion coefficient matrix; lattice gas.

## INTRODUCTION

Let $\mathbf{Z}$ denote the integer lattice. The zero-range-exclusion process that we are to study is a Markov process on the state space $\mathscr{X} := \mathbf{Z}_+^{\mathbf{Z}}$, the infinite product of $\mathbf{Z}_+ := \{0, 1, 2,...\}$. Denote by $\eta = (\eta_x, x \in \mathbf{Z})$ a generic element of $\mathscr{X}$, and define

$$\xi_x = \mathbf{1}(\eta_x > 0),$$

where $\mathbf{1}(A)$ stands for the indicator of an event $A$. The process is regarded as a gas of particles carrying energy. The site $x$ is occupied by a particle if $\xi_x = 1$ and vacant otherwise. Each particle has energy, represented by $\eta_x$, which takes discrete values $1, 2,...$ . A particle at site $x$ jumps to a nearest neighbor site $y = x \pm 1$ at rate $c_{\rm ex}(\eta_x)$ if $y$ is vacant, where $c_{\rm ex}$ is a positive function of $k = 1, 2,...$ . Between two adjacent particles the energies are

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0022-4715/04/0600-1423/0 

transferred unit by unit according to the same stochastic rule as that of the zero range processes with a rate function $c_{zr}$. This dynamics has two conserved quantities, the number of particles and the total energy. It is not a gradient system. We shall assume that both $c_{ex}$ and $c_{zr}$ grow in the linear order, a condition under which we can prove a suitable estimate for spectral gaps of generators of the localized processes (cf. ref. 15). We shall consider the time-evolution process of fluctuation fields in equilibrium under the parabolic scaling and prove that it converges to those of an infinite dimensional Ornstein–Uhlenbeck process. It is straightforward to extend these results to the multidimensional model: the restriction to one dimension is only for simplicity of exposition and notation.

The convergence of fluctuation processes under parabolic scaling has been dealt with for various models (cf. refs. 2–4, 6, 13, 17, etc.). Among them the results for non-gradient models are all based on Varadhan's proof of the hydrodynamic limit for a non-gradient Ginzburg–Landau model (cf. ref. 20). One of the principal ingredients in that proof is the fundamental structure theorem of the quadratic form which is a limit of space-time variance relative to the localized dynamics. It may well be said that the convergence result on equilibrium fluctuations is an almost immediate consequence of this theorem, possibly except for technical details that may depend on the models. For the present model the space of spin values is not compact: the energy current involves a term of quadratic growth in energy, which the relative entropy alone can not control; in the Dirichlet form the square of the gradient is multiplied by an unbounded factor of $c_{zr}$ or $c_{ex}$; a spectral gap estimate that is available for the moment is not uniform in two conserved quantities although of proper order relative to the size of physical space of the local system. These together cause difficulty for adapting the argument of ref. 20 to prove hydrodynamic limit for the present model. Of the equilibrium fluctuations, however, we can solve the problem, which is more tractable than that of the density fields in non-equilibrium. For the reasons advanced above the manner of how to control unbounded spins is somewhat different from one for non-gradient Ginzburg–Landau model which is only non-gradient model having unbounded spins among those that have been previously studied. For the proof we shall adapt the corresponding ones of refs. 8 and 11, albeit its basic idea is the same as that of ref. 20. Recently equilibrium fluctuations are investigated for other models in which there arise problems to be solved specific to the models.[1, 5, 16]

The diffusion matrix $D = D(p, \rho)$ that is conceived to be a diffusion coefficient is defined by the variational formula due to Varadhan[20] (see (5) in Section 1). The definition has to be justified. It is motivated by the fundamental structure theorem mentioned above that we need establish for the present model. It relates $D$ with the (microscopic) current via microscopic Fick's law (fluctuation dissipation equation) that equates the current with the density gradient multiplied by $D$ from the left plus a fluctuation term. In a heuristic level this leads one to expect that the hydrodynamic equation for the limit densities $p = p(t, \theta)$ and $\rho = \rho(t, \theta)$ of our system should be

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ \rho \end{pmatrix} = \frac{\partial}{\partial \theta} D(p, \rho) \frac{\partial}{\partial \theta}\begin{pmatrix} p \\ \rho \end{pmatrix},$$

of which the validity has not been established.

If our process (on $\mathbf{Z}$) is in the equilibrium of particle and energy densities $p$ and $\rho$, respectively and $\eta(t) = (\eta_x(t))_{x \in \mathbf{Z}}$ denotes a sample configuration at time $t$ of the process, the fluctuation field scaled with large parameter $N$ is the $\mathbf{R}^2$-valued measure on $\mathbf{R}$ given by

$$\begin{pmatrix} Y^P_{t,N}(d\theta) \\ Y^E_{t,N}(d\theta) \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{x \in \mathbf{Z}} \begin{pmatrix} \xi_x(N^2 t) - p \\ \eta_x(N^2 t) - \rho \end{pmatrix} \delta_{x/N}(d\theta) \qquad (\theta \in \mathbf{R})$$

where $\xi_x(t) = \mathbf{1}(\eta_x(t) > 0)$ and $\delta_{x/N}$ is the delta measure carrying unit mass at $x/N$. Our main result in this paper asserts that $Y^N_t := (Y^P_{t,N}, Y^E_{t,N})$ converges in law to an infinite dimensional Ornstein–Uhlenbeck process, $Y_t = (Y^P_t, Y^E_t)$ say, which is a solution of the stochastic differential equation

$$\begin{pmatrix} dY^P_t \\ dY^E_t \end{pmatrix} = D \frac{\partial^2}{\partial \theta^2} \begin{pmatrix} Y^P_t \\ Y^E_t \end{pmatrix} dt + \sqrt{2D\chi} \frac{\partial}{\partial \theta} \begin{pmatrix} dw^1_t \\ dw^2_t \end{pmatrix},$$

where $w^1 = (w^1_t(\theta))_{t \geqslant 0}$ is an infinite dimensional mean zero Brownian motion with the variance functional $t \|J\|^2_{L^2(\mathbf{R})}$ $(J \in C^\infty_0(\mathbf{R}))$, $w^2$ an independent copy of $w^1$ and $\chi = \chi(p, \rho)$ the covariance matrix for the pair $(\xi_0, \eta_0)$.

## 1. MODEL AND RESULTS

To give a precise definition of the process we introduce some notations. Let $b = (x, y)$ be an oriented bond of $\mathbf{Z}$, namely $x$ and $y$ are nearest neighbor sites of $\mathbf{Z}$, and $(x, y)$ stands for an ordered pair of them. Define the exclusion operator $\pi^{\mathrm{ex}}_b$ and zero-range operator $\pi^{\mathrm{zr}}_b$ attached to $b$ which acts on the space of functions $f$ of $\eta \in \mathscr{X}$ by

$$\pi^{\mathrm{ex}}_b f(\eta) = f(S^b_{\mathrm{ex}}\eta) - f(\eta) \qquad \text{and} \qquad \pi^{\mathrm{zr}}_b f(\eta) = f(S^b_{\mathrm{zr}}\eta) - f(\eta)$$

where $S^b_{\mathrm{ex}}$ and $S^b_{\mathrm{zr}}$ are transformations on $\mathscr{X}$ defined as follows: if $\xi_x = 1$ and $\xi_y = 0$, then

$$(S^b_{\mathrm{ex}}\eta)_z = \begin{cases} \eta_y, & \text{if} \quad z = x, \\ \eta_x, & \text{if} \quad z = y, \\ \eta_z, & \text{otherwise;} \end{cases}$$

and if $\eta_x \geqslant 2$ and $\xi_y = 1$, then

$$(S^b_{\mathrm{zr}}\eta)_z = \begin{cases} \eta_x - 1, & \text{if} \quad z = x, \\ \eta_y + 1, & \text{if} \quad z = y, \\ \eta_z, & \text{otherwise;} \end{cases}$$

and in the remaining cases of $\eta$, both $S^b_{\mathrm{ex}}\eta$ and $S^b_{\mathrm{zr}}\eta$ are set to be $\eta$, namely

$$S^b_{\mathrm{ex}}\eta = \eta \qquad \text{if} \quad \xi_x(1-\xi_y) = 0,$$
$$S^b_{\mathrm{zr}}\eta = \eta \qquad \text{if} \quad \mathbf{1}(\eta_x \geqslant 2)\, \xi_y = 0.$$

Let $c_{\mathrm{ex}}$ and $c_{\mathrm{zr}}$ be two nonnegative functions on $\mathbf{Z}_+$ and define

$$L_b = c_{\mathrm{ex}}(\eta_x)\, \pi^{\mathrm{ex}}_b + c_{\mathrm{zr}}(\eta_x)\, \pi^{\mathrm{zr}}_b.$$

Let $\Lambda$ be a finite interval of $\mathbf{Z}$. The transformations $S^b_{\mathrm{zr}}$, $S^b_{\mathrm{ex}}$ and the operators $\pi^{\mathrm{ex}}_b$, $\pi^{\mathrm{zr}}_b$, $L_b$ naturally act on the local configuration space $\mathbf{Z}^\Lambda_+$ and the functions on it, respectively, provided $b$ is an oriented bond of $\Lambda$. Let $\Lambda^*$ denote the set of all oriented bonds in $\Lambda$:

$$\Lambda^* = \{b = (x, y)\colon x, y \in \Lambda, |x - y| = 1\}.$$

Then the infinitesimal generator $L_\Lambda$ of our Markovian particle process on $\mathbf{Z}^\Lambda_+$, which we shall often call a lattice gas on $\Lambda$ for convenience, is given by

$$L_\Lambda = \sum_{b \in \Lambda^*} L_b.$$

The lattice gas on $\mathbf{Z}$ will be a limit of those on finite lattices and introduced later. It is assumed that for some positive constant $a_0$, $c_{\mathrm{ex}}(k) \geqslant a_0$ for $k \geqslant 1$ and $c_{\mathrm{zr}}(k) \geqslant a_0$ for $k \geqslant 2$. This especially implies that the lattice gas on $\Lambda$ with both the number of particles and the total energy being specified is ergodic. We call our lattice gas *a zero-range-exclusion process* whether its physical space is $\Lambda$ or $\mathbf{Z}$. For sake of convenience we set

$$c_{\mathrm{ex}}(0) = 0 \qquad \text{and} \qquad c_{\mathrm{zr}}(0) = c_{\mathrm{zr}}(1) = 0.$$

The functions $c_{\mathrm{ex}}$ and $c_{\mathrm{zr}}$ are further supposed to satisfy the following conditions:

$$|c_{\mathrm{zr}}(k)-c_{\mathrm{zr}}(k+1)| \leqslant a_1 \qquad \text{for all} \quad k \geqslant 1; \tag{1}$$

$$c_{\mathrm{zr}}(k)-c_{\mathrm{zr}}(l) \geqslant a_2 \qquad \text{whenever} \quad k \geqslant l+k_0; \tag{2}$$

$$a_3 k \leqslant c_{\mathrm{ex}}(k) \leqslant a_4 k \qquad \text{for all} \quad k \geqslant 1. \tag{3}$$

where $a_1, a_2, a_3$, and $k_0$ are positive constants. We shall sometimes write $\pi^{\mathrm{ex}}_{x,y}$, $S^{x,y}_{\mathrm{ex}}$, $L_{x,y}$, etc. for $\pi^{\mathrm{ex}}_b$, $S^b_{\mathrm{ex}}$, $L_b$, etc. if $b=(x,y)$.

*Grand Canonical Measures.* For a pair of constants $0<p<1$ and $\rho>p$ let $\nu_{p,\rho}=\nu^{\mathbf{Z}}_{p,\rho}$ denote the product probability measure on $\mathscr{X}$ whose marginal laws are given by

$$\nu_{p,\rho}(\{\eta\colon \eta_x=l\}) := \begin{cases} 1-p & \text{if} \quad l=0,\\ \dfrac{p}{Z_{\alpha(p,\rho)}} & \text{if} \quad l=1,\\ \dfrac{p}{Z_{\alpha(p,\rho)}}\,\dfrac{(\alpha(p,\rho))^{l-1}}{c_{\mathrm{zr}}(2)\,c_{\mathrm{zr}}(3)\cdots c_{\mathrm{zr}}(l)} & \text{if} \quad l\geqslant 2,\end{cases}$$

for all $x\in\mathbf{Z}$. Here $Z_\alpha$ is the normalizing constant:

$$Z_\alpha := 1+\sum_{l=2}^{\infty}\frac{\alpha^{l-1}}{c_{\mathrm{zr}}(2)\,c_{\mathrm{zr}}(3)\cdots c_{\mathrm{zr}}(l)}$$

and $\alpha(p,\rho)$ is a positive constant depending on $p$ and $\rho$ and uniquely determined by the relation

$$E^{\nu_{p,\rho}}[\eta_x]=\rho,$$

where $E^{\nu_{p,\rho}}$ denotes the expectation under the law $\nu_{p,\rho}$. Under the condition imposed on $c_{\mathrm{zr}}$ the function $\alpha(p,\rho)$ is well defined. Clearly $E^{\nu_{p,\rho}}[\xi_x]=p$.

For a finite interval $\Lambda$ of $\mathbf{Z}$ we denote by $\nu^{\Lambda}_{p,\rho}$ the projection of $\nu_{p,\rho}$ to $\mathbf{Z}^{\Lambda}_{+}$. The lattice gas on $\Lambda$ is reversible relative to the measures $\nu^{\Lambda}_{p,\rho}$ (namely $L_\Lambda$ is symmetric relative to each of them) as is easily shown (see (4) and (5) below), and these measures play the role of grandcanonical Gibbs measures for the lattice gas.

*The Operator $\Gamma_b$.* It is convenient to introduce the transformations

$$S^b\eta = \begin{cases} S^b_{\mathrm{ex}}\eta & \text{if} \quad \xi_y=0,\\ S^b_{\mathrm{zr}}\eta & \text{if} \quad \xi_y=1,\end{cases}$$

and the operators

$$\Gamma_b = \xi_x \pi_b^{\mathrm{ex}} + \mathbf{1}(\eta_x \geqslant 2)\, \pi_b^{\mathrm{zr}},$$

where $b = (x, y)$. Alternatively the latter may also be defined by

$$\Gamma_b f(\eta) = f(S^b \eta) - f(\eta).$$

Let $\tau_x \eta$ be the configuration $\eta \in \mathscr{X}$ viewed from $x$, namely $(\tau_x \eta)_y = \eta_{x+y}$. We let it also act on a function $f$ of $\eta$ according to $\tau_x f(\eta) = f(\tau_x \eta)$. Setting

$$c_{01}(\eta) = c_{\mathrm{ex}}(\eta_0)(1-\xi_1) + c_{\mathrm{zr}}(\eta_0)\, \xi_1;$$

$$c_{10}(\eta) = c_{\mathrm{ex}}(\eta_1)(1-\xi_0) + c_{\mathrm{zr}}(\eta_1)\, \xi_0;$$

and $c_{x,x+1} = \tau_x c_{01}$, $c_{x+1,x} = \tau_x c_{10}$, we can write

$$L_b = c_b \Gamma_b.$$

The Dirichlet form for $L_\Lambda$ under the grandcanonical measure $\nu_{p,\rho}$ is given by

$$\mathscr{D}_{p,\rho}^{\Lambda}\{f\} = \frac{1}{2} \sum_{b \in \Lambda^*} E^{\nu_{p,\rho}}[|\Gamma_b f|^2\, c_b] \tag{4}$$

for suitable functions $f$.

*Diffusion Coefficient Matrix.* To define the diffusion coefficient matrix we need to introduce some more notations. A function $f$ on $\mathscr{X}$ is called local if $f$ depends only on a finite number of coordinates of $\eta \in \mathscr{X}$. For the sake of convenience we consider only local functions which is real valued and of at most polynomial growth. To be precise let $\Lambda$ be a finite subset of $\mathbf{Z}$ and $\mathscr{F}_\Lambda$ the space of real functions on $\mathbf{Z}_+^\Lambda$ such that

$$|f(\eta)| \leqslant K \sum_{x \in \Lambda} \eta_x^K$$

for some positive integer $K$. An element of $\mathscr{F}_\Lambda$ may be regarded as a function on $\mathscr{X}$ in self-evident way. Let $\Lambda(n)$ denote the interval $\{-n,..., n\} \subset \mathbf{Z}$ and $\mathscr{F}_c$ the union $\bigcup_{n=1}^{\infty} \mathscr{F}_{\Lambda(n)}$. For $f \in \mathscr{F}_c$ we use the symbol $\tilde{f}$ to represent the formal sum $\sum_x \tau_x f$. It has meaning if $\Gamma_{01} := \Gamma_{0,1}$ acts on it in such a manner that

$$\Gamma_{01} \tilde{f} = \sum_x \Gamma_{01} \tau_x f = \sum_x \tau_x \Gamma_{x,x+1} f,$$

where the infinite sums are actually finite sums.

Let $0 < p < 1$, $\rho > p$. Let $\chi(p, \rho)$ denote the covariance matrix of $\xi_0$ and $\eta_0$ under $\nu_{p,\rho}$:

$$\chi(p, \rho) = \begin{pmatrix} (1-p)\,p & (1-p)\,\rho \\ (1-p)\,\rho & E^{\nu_{p,\rho}}\,|\eta_0 - \rho|^2 \end{pmatrix}.$$

For $\underline{f} = (f_1, f_2)^T \in \mathscr{F}_c \times \mathscr{F}_c$ ($T$ indicates the transpose) let

$$\hat{c}(p, \rho; \underline{f}) = (\hat{c}^{i,j}(p, \rho; \underline{f}))_{1 \leqslant i, j \leqslant 2}$$

be a $2 \times 2$ symmetric matrix whose quadratic form is given by

$$\underset{\sim}{\alpha} \cdot \hat{c}(p, \rho; \underline{f})\, \underset{\sim}{\alpha} = E^{\nu_{p,\rho}}[(\Gamma_{01}\{\alpha(\xi_0 + \tilde{f}_1) + \beta(\eta_0 + \tilde{f}_2)\})^2\, c_{01}]$$

where $\underset{\sim}{\alpha} = (\alpha, \beta)^T$, a two-dimensional real column vector, and $\cdot$ indicates the inner product in $\mathbf{R} \times \mathbf{R}$; also define a $2 \times 2$ symmetric matrix $\hat{c}(p, \rho)$ via the variational formula:

$$\underset{\sim}{\alpha} \cdot \hat{c}(p, \rho)\, \underset{\sim}{\alpha} = \inf_{\underline{f} \in \mathscr{F}_c \times \mathscr{F}_c} \underset{\sim}{\alpha} \cdot \hat{c}(p, \rho; \underline{f})\, \underset{\sim}{\alpha}.$$

Then the diffusion coefficient matrix is defined by

$$D(p, \rho) = \hat{c}(p, \rho)\, \chi^{-1}(p, \rho), \tag{5}$$

where $\chi^{-1}(p, \rho)$ denotes the inverse matrix of $\chi(p, \rho)$. The matrix $\hat{c}(p, \rho)$, as well as $\chi^{-1}(p, \rho)$, is positive definite. This does not imply that $D(p, \rho)$ is positive definite but implies that both of two eigenvalues of $D(p, \rho)$ are positive and if they coincide, then $D(p, \rho)$ is a constant times the unit matrix.

***The Lattice Gas on* $\mathbf{Z}$.** Now we consider an infinite particle system on the whole lattice $\mathbf{Z}$. Its formal generator is

$$Lf(\eta) = \sum_{b \in \mathbf{Z}^*} c_b(\eta)\, \Gamma_b f(\eta), \qquad f \in \mathscr{F}_c.$$

The following theorem is a consequence from a standard theory of Markov semigroups (cf., eg., ref. 7). The parameters $p$ and $\rho$ are supposed to be given as before.

**Theorem A.** (i) The operator $L$ defined on $\mathscr{F}_c$ as above is closable and its smallest closed extension, denoted by $\mathscr{L}$, in the space $L^2(\nu_{p,\rho}, \mathscr{X})$ generates a strongly continuous Markov semigroup on $L^2(\nu_{p,\rho}, \mathscr{X})$.

(ii) Denote by $S(t)$, $t \geqslant 0$ this semigroup, and by $S_K(t)$ the semigroup on $L^2(\nu_{p,\rho}^{\Lambda(K)}, \mathscr{X}_{\Lambda(K)})$ generated by $L_{\Lambda(K)}$. Then

$$\underset{K \to \infty}{\text{s-lim}}\ S_K(t)\, f = S(t)\, f, \qquad f \in \mathscr{F}_c$$

uniformly in $t$ on each finite interval. Here $S_K(t)$ acts on $f(\eta), \eta \in \mathscr{X}$ by regarding $f$ as a function of $\eta|_{\Lambda(K)} \in \mathbf{Z}_+^{\Lambda(K)}$ with the other coordinates being frozen; the strong limit, s-lim, means the convergence in the $L^2$-norm.

Let the configuration space $\mathscr{X}$ be endowed with the product topology. It may then be regarded as a complete and separable metric space. The processes generated by $L_{\Lambda(K)}$ may be considered as processes on $\mathscr{X}$ with $\eta_y$, $y \notin \Lambda(K)$ being frozen as for $S_K(t)$ in Theorem A. Let these process start with the equilibrium state $\nu_{p,\rho}$. Then we have a sequence of probability laws on the Skorohod space $D([0, T], \mathscr{X})$, which is tight as is easily assured. As a limit process we have a stationary Markov process on $\mathscr{X}$ such that its transition law is given by $S(t)$ and its marginal law is $\nu_{p,\rho}$ and that its sample paths are continuous from the right and have limits from the left with probability one. We denote the probability law of the process by $P_{\text{eq}(p,\rho)}$ and the corresponding expectation by $E_{\text{eq}(p,\rho)}$, and write $\eta(t), t \geqslant 0$ for a generic sample path of the process.

*The Fluctuation Fields.* Define two random functionals $Y_{t,N}^P$ and $Y_{t,N}^E$ by

$$Y_{t,N}^P(J) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbf{Z}} J(x/N)(\xi_x(N^2 t) - p),$$

$$Y_{t,N}^E(J) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbf{Z}} J(x/N)(\eta_x(N^2 t) - \rho),$$

where $J \in C_0^\infty(\mathbf{R})$ (namely $J$ is a compactly supported smooth function) and

$$\xi_x(t) = \mathbf{1}(\eta_x(t) > 0).$$

The random functionals $Y_{t,N}^P$ and $Y_{t,N}^E$ are fluctuation fields of particle density and energy density, respectively, and may be understood to be processes taking values in the space of, eg., signed measures. Its limiting process will take values in the space of tempered distribution. Let $\mathbf{H}^{(-\lambda)}$ ($\lambda \in \mathbf{R}$) denote the completion of the space of rapidly decreasing functions $\phi(\theta)$ with the norm $\|\phi\|_{(-\lambda)} := \|(\theta^2 - (d^2/d\theta^2))^{-\lambda/2}\, \phi\|_{L^2}$. Then the limit process, $Y_t = (Y_t^P, Y_t^E)^T$ say, will be a $\mathbf{H}^{(-\lambda)}$-valued process (with $\lambda > 2$)

whose sample path is in the space of continuous trajectories $C([0,T],\mathbf{H}^{(-\lambda)})$. The main theorem of this paper is stated as follows.

**Theorem 1.** Let $\lambda > 2$. Then under the equilibrium measure $P_{\text{eq}(p,\rho)}$ the sequence of $\mathbf{H}^{(-\lambda)}$-valued processes $Y_t^N := (Y_{t,N}^P, Y_{t,N}^E)^T$ weakly converges to the Ornstein–Uhlenbeck process in $\mathbf{H}^{(-\lambda)}$ which is characterized by the following stochastic integral equation

$$Y_t(J) = Y_0(J) + \int_0^t D(p,\rho)\, Y_s(J'')\, ds + \sqrt{2\hat{c}(p,\rho)}\, w_t(J').$$

Here $w_t = (w_t^1, w_t^2)^T$ with $w_t^1$ being a $\mathbf{H}^{(-\lambda+1)}$-valued Wiener process whose variance is $E\,|w_t^1(J)|^2 = t\,\|J\|_{L^2(\mathbf{R})}^2$ and $w_t^2$ being an independent copy of $w_t^1$ and $Y_t(J) = (Y_t^P(J), Y_t^E(J))^T$.

**Remark.** Define for $k \geqslant 0$

$$h_k(\theta) = (\sqrt{\pi}\, 2^k k!)^{-1/2}\, e^{\theta^2/2} (-d/d\theta)^k\, e^{-\theta^2} \qquad (\theta \in \mathbf{R}).$$

Then $\{h_k\}$ constitutes an orthonormal basis of eigen-functions of the self-adjoint operator $\theta^2 - (d/d\theta)^2$ in $L^2(\mathbf{R})$ with eigen-values $2k+1$, so that the norm $\|\cdot\|_{(-\lambda)}$ may be expressed as

$$\|\phi\|_{(-\lambda)} = \Big[\sum |(\phi, h_k)_{L^2}|^2\, (2k+1)^{-\lambda}\Big]^{-1/2}.$$

We also have the formulas $2\theta h_k = \sqrt{2k}\, h_{k-1} + \sqrt{2k+2}\, h_{k+1}$ and $2h_k' = \sqrt{2k}\, h_{k-1} - \sqrt{k+1}\, h_{k+1}$ and the bound $\|h_k\|_{L^1} \leqslant 4(k+1)^{1/4}$, which are often useful for showing the tightness of $\mathbf{H}^{(-\lambda)}$-valued processes. (Cf. eg., ref. 9.)

*The Space-Time Correlation of $(\xi_x(t), \eta_x(t))$.* Let $K_D$ denote the fundamental solution to the initial value problem for the heat equation $\frac{\partial}{\partial t}\underline{u} = D^T \frac{\partial^2}{\partial \theta^2}\underline{u}$ ($\underline{u} = \underline{u}(\theta, t)$ and $D^T$ is the transpose of $D$) and $U_t$ a matrix of operators which represents the corresponding convolution semigroup: $U_t \underline{J}(\theta) = \int_{-\infty}^{\infty} K_D(t, \theta-\theta')\, \underline{J}(\theta')\, d\theta'$. Then the covariance function of the limit process $Y_t$ in Theorem 1 is given by

$$E[(Y_0, \underline{J}_1)(Y_t, \underline{J}_2)] = \int_{\mathbf{R}} \chi(p,\rho)\, U_t \underline{J}_2(\theta) \cdot \underline{J}_1(\theta)\, d\theta,$$

where $E$ denotes the expectation for the limit process.

Let $\Sigma(x,t)=\Sigma_{p,\rho}(x,t)$ be the symmetric $2\times 2$ matrix, depending on $(x,t)\in \mathbf{Z}\times[0,\infty)$, whose quadratic form is

$$\underline{\alpha}\cdot\Sigma(x,t)\,\underline{\alpha}$$
$$=E_{\mathrm{eq}(p,\rho)}[\{\alpha(\xi_0(0)-p)+\beta(\eta_0(0)-\rho)\}\{\alpha(\xi_x(t)-p)+\beta(\eta_x(t)-\rho)\}].$$

Since $v_{p,\rho}$ is invariant under the translation, $\Sigma(x,t)$ is the covariance matrix of $(\xi_x(s),\eta_x(s))$ and its space-time translation $(\xi_{x+y}(s+t),\eta_{x+y}(s+t))$. Hence if we define

$$R(x,t):=\chi^{-1}(p,\rho)\,\Sigma(x,t),$$

then $R(x-y,t-s)$ is the space-time correlation coefficient of $(\xi_x(t),\eta_x(t))$.

The next theorem, which relates $R$ with $K_D$ somehow directly, is deduced from Theorem 1 (cf. ref. 19 for proof).

**Theorem 2.** For $\underline{J}=(J^1,J^2)^T\in C_0^\infty(\mathbf{R})\times C_0^\infty(\mathbf{R})$

$$\lim_{N\to\infty}\sum_{x\in\mathbf{Z}}R(x,N^2t)\,\underline{J}(x/N)=\int_{-\infty}^{\infty}K_D(\theta,t)\,\underline{J}(\theta)\,d\theta.$$

One may expect that the diffusion coefficient matrix would be given by

$$D_T=\lim_{t\to\infty}\frac{1}{2t}\sum_{x\in\mathbf{Z}}x^2R(x,t),$$

and this would lead to the Green–Kubo formula for $D$ (cf. Spohn[18]). However, we know too little on the behavior of the tail of $R$ to verify these assertions.

To conclude this section we point out that in the derivation of the hydrodynamic limit to the present model there arises difficulty due to unboundedness of spin values. While the marginal of our grandcanonical measure is roughly Poisson, the energy current from the site 0 to the site 1, being given by

$$w_{01}^E:=c_{\mathrm{ex}}(\eta_0)(1-\xi_1)\,\eta_0+c_{\mathrm{zr}}(\eta_0)\,\xi_1-c_{\mathrm{ex}}(\eta_1)(1-\xi_0)\,\eta_1-c_{\mathrm{zr}}(\eta_1)\,\xi_0,$$

involves the term $c_{\mathrm{ex}}(\eta_0)\,\eta_0$ that is bounded below by $\delta\eta_0^2$ ($\delta>0$) and cannot be controlled by the grandcanonical measure via the entropy inequality as in the case of Ginzburg–Landau model, the logarithm of the Poisson density function being of the order $O(\eta_0\log\eta_0)$. We can show that the expectation of the space-time average of the variable $[(\xi_{x-1}+\xi_{x+1})\,\eta_x^3+\eta_x^2](N^2t)$ for processes on the discrete torus $\mathbf{Z}/N\mathbf{Z}$ is bounded as $N$ tends to infinity but do not know of the corresponding uniform integrability (that is crucial for a standard truncation argument). Even if we assume that $c_{\mathrm{zr}}$ and $c_{\mathrm{ex}}$ are constant, the situation does not much change: we would be able to show results on the central limit theorem variance analogous to those given in Section 2 and accordingly on the equilibrium fluctuation, but do not know of uniform integrability of $\eta_x(tN^2)$; moreover $\mathscr{L}_{\{0,1\}}^{-1}\{\eta_0 - \eta_1\}$ grows as $(\eta_0 + \eta_1)^2$ on the event $\xi_0 = \xi_1 = 1$ (cf. Lemma 4), which causes an additional difficulty for derivation of the hydrodynamic limit.

The rest of the paper is organized as follows. In Section 2 we shall introduce a quadratic form which is defined as the limit of central limit theorem variances for time evolutions of the local processes and state the structure theorem for the quadratic form, which is fundamental for the proof of Theorem 1. While its proof is outlined there, the details are postponed to Sections 4 and 5. The proof of Theorem 1 will be given in Section 3 where results of Section 2 (Theorem 3 and Lemma 7) are applied. In Section 4 we shall introduce a space of functions on $\mathscr{X}$ called closed forms and determine the structure of the space, an equivalent of the structure theorem of Section 2. In Section 5 we give a proof of a lemma used in Section 4.

## 2. CENTRAL LIMIT THEOREM VARIANCE

This section will be divided into three subsections. The main result of it is given in the second subsection while its proof is outlined in the third.

### 2.1. Canonical Measures and Reversibility Relations

The canonical measure for the configurations on $\Lambda(n)$ with the number of particles $m$ and the total energy $E$ is the conditional law

$$P_{n,m,E}[\,\cdot\,] = \frac{\nu_{p,\rho}^{\Lambda(n)}(\,\cdot\, \cap \{|\xi|_{\Lambda(n)} = m,\, |\eta|_{\Lambda(n)} = E\})}{\nu_{p,\rho}(\{|\xi|_{\Lambda(n)} = m,\, |\eta|_{\Lambda(n)} = E\})}.$$

Here $|\xi|_\Lambda = \sum_{x\in\Lambda} \xi_x$ and $|\eta|_\Lambda = \sum_{x\in\Lambda} \eta_x$. The reversibility is equivalent to the set of the detailed balance conditions

$$c_{\mathrm{zr}}(\eta_x)\, \xi_y P_{n,m,E}\{\eta\} = c_{\mathrm{zr}}(\eta_y + 1)\, \mathbf{1}(\eta_x \geqslant 2)\, P_{n,m,E}\{S_{\mathrm{zr}}^{x,y}\eta\} \tag{6}$$

and

$$P_{n,m,E}\{\eta\} = P_{n,m,E}\{S_{\mathrm{ex}}^{x,y}\eta\}, \tag{7}$$

both of which are valid for any $n, m, E \in \mathbf{Z}_+$ $(m \leqslant n, E)$, for any two sites $x, y \in \Lambda(n)$ and for any configuration $\eta$ on $\Lambda(n)$. Here $P\{\eta\}$ denotes the $P$-measure of a single point set $\{\eta\}$. From (6) it follows that for any functions $f$ and $g$ of $\eta$,

$$\langle c_{\mathrm{zr}}(\eta_x)\, \xi_y\, f(S_{\mathrm{zr}}^{x,y}\eta)\, g(\eta)\rangle_{n,m,E} = \langle c_{\mathrm{zr}}(\eta_y)\, \xi_x\, f(\eta)\, g(S_{\mathrm{zr}}^{y,x}\eta)\rangle_{n,m,E};$$

the analogue for $S_{\mathrm{ex}}^{x,y}$ would be obvious. Here, and throughout the rest of this paper, $\langle\cdot\rangle_{n,m,E}$ indicates the expectation by $P_{n,m,E}$. The Dirichlet form on the configuration space $\mathscr{X}_{n,m,E} := \{\eta \in \mathbf{Z}_+^{\Lambda(n)} : |\xi|_{\Lambda(n)} = m, |\eta|_{\Lambda(n)} = E\}$ is accordingly given by

$$\begin{aligned}\mathscr{D}_{n,m,E}\{f\} &:= -\langle f L_{n,m,E} f\rangle_{n,m,E}\\ &= \sum_{b \in \Lambda^*(n)} \mathscr{D}_{n,m,E}^{b}\{f\},\end{aligned}$$

where $\Lambda^*(n) = (\Lambda(n))^*$, $L_{n,m,E}$ denotes the operator $L_{\Lambda(n)}$ restricted to the space of functions $f$ on $\mathscr{X}_{n,m,E}$ and

$$\mathscr{D}_{n,m,E}^{b}\{f\} = \tfrac{1}{2}\langle(\Gamma_b f)^2\, c_b\rangle_{n,m,E}.$$

From the reversibility relations (6) and (7) it follows that

$$\langle f L_{01}\, g\rangle_{n,m,E} = \langle L_{10} f \cdot g\rangle_{n,m,E}$$

(recall $L_b = c_b \Gamma_b$) and that

$$\langle f \cdot (L_{01} + L_{10})\, g\rangle_{n,m,E} = \langle(\Gamma_{01} f)(\Gamma_{01}\, g)\, c_{01}\rangle_{n,m,E}; \tag{8}$$

in particular, for the associated bilinear form,

$$\mathscr{D}_{n,m,E}^{01}(f, g) = \mathscr{D}_{n,m,E}^{10}(g, f).$$

## 2.2. Central Limit Theorem Variance

We introduce a function space on which the central limit theorem variance is well defined. The numbers $p$ and $\rho$ are fixed so that $0 < p < 1$ and $\rho \geqslant p$ unless otherwise specified.

**Definition 1.** Let $\mathscr{G}$ denote the linear space of all functions $h \in \mathscr{F}_c$ of the form

$$L_I H := \sum_{b \in I^*} L_b H = h \tag{9}$$

where $I = I(h)$ is a (finite) interval of $\mathbf{Z}$ and $H$ is a local function such that $I^*$ is not void and

$$\Gamma_b H \in \mathscr{F}_c \qquad \text{for all} \quad b \in I^*. \tag{10}$$

(We can and shall choose $H$ such that the expectation of $H$ by each canonical measure on $I$ vanishes.) For a pair $I = I(h)$ and $H$ chosen as above, we write $H = (L_{I(h)})^{-1} h$.

If $h \in \mathscr{F}_c$ satisfies

$$E^{\nu_{p,\rho}}[h; |\xi|_I = m, |\eta|_I = E \mid \mathscr{F}_{\mathbf{Z}\setminus I}] = 0 \qquad \text{for all} \quad m \leqslant \sharp I \quad \text{and} \quad E \geqslant m$$

($\sharp A$ stands for the cardinality of a set $A$), then it admits a representation (9), whereas the condition (10) still remains to be verified. (Recall that (10) means that $\Gamma_b H$ grows with at most polynomial order.) The linear space

$$\{Lf \colon f \in \mathscr{F}_c\}$$

is obviously included in $\mathscr{G}$. The currents $w_{01}^P$, $w_{01}^E$ which are defined by

$$w_{01}^P = -L_{\{0,1\}}\{\xi_0\} \qquad \text{and} \qquad w_{01}^E = -L_{\{0,1\}}\{\eta_0\}$$

are also in $\mathscr{G}$ (notice that $L_{\{0,1\}} = L_{01} + L_{10}$): the requirements are satisfied with $I = \{0, 1\}$; and $H = -\xi_0$ or $H = -\eta_0$. It may be remarked that explicit forms of $w_{01}^P$ or $w_{01}^E$ are not important at all for our analysis in this paper.

For $h$ and $g$ from $\mathscr{G}$, define (for $n$ large enough)

$$V_{n,m,E}(h, g) = \frac{1}{2n} \left\langle \sum_{|x| < n'} \tau_x h \cdot (-L_{n,m,E})^{-1} \sum_{|x| < n'} \tau_x g \right\rangle_{n,m,E}.$$

Here $n'$ is a positive integer such that $n - n'$ is a constant (independent of $n$) chosen so that both sums under the expectation are $\mathscr{F}_{\Lambda(n)}$-measurable but otherwise may be arbitrarily determined. Notice that $(-L_{n,m,E})^{-1}$ is well defined as a transformation on the space $\{F \in C(\mathscr{X}_{n,m,E}) : \langle F \rangle_{n,m,E} = 0\}$. The following theorem, originally discovered for a Ginzburg–Landau model,[20] reveals a structure of the space $\mathscr{G}$ equipped with the quadratic form of $V = \lim V_{n,m,E}$ (see also Lemma 10 in the next section for a related result).

**Theorem 3.** For every $h, g \in \mathscr{G}$, there exists a following limit

$$\lim_{(m/2n,\, E/2n) \to (p,\rho)} V_{n,m,E}(h, g),$$

where the limit is taken in such a way that $n$, $m$ and $E$ are sent to infinity so that $m/2n \to p$ and $E/2n \to \rho$. This limit makes a bilinear form on $\mathscr{G}$ and is denoted by

$$V^{p,\rho}(h, g).$$

Let $\mathscr{F}_{c,b}$ be the set of all bounded functions from $\mathscr{F}_c$. Then the subspace

$$\mathscr{G}_o := \{\alpha w_{01}^P + \beta w_{01}^E - Lf : \alpha, \beta \in \mathbf{R}, f \in \mathscr{F}_{c,b}\}$$

is dense in $\mathscr{G}$ with respect to the quadratic form $V^{p,\rho}\{h\} := V^{p,\rho}(h, h)$.

Theorem 3 says that every $h \in \mathscr{G}$ can be approximated by an element of $\mathscr{G}_o$ in the metric $\sqrt{V^{p,\rho}}$ as accurately as one needs. We shall apply this to the gradients

$$\nabla^-\xi := \xi_0 - \xi_1 \qquad \text{and} \qquad \nabla^-\eta := \eta_0 - \eta_1.$$

To this end we need the following lemma.

**Lemma 4.** Both $\nabla^-\xi$ and $\nabla^-\eta$ are in $\mathscr{G}$ with $I = \{0, 1\}$. If $H^P$ and $H^E$ denote the corresponding $H$'s (namely, $L_I H^P = \nabla^-\xi$ and $L_I H^E = \nabla^-\eta$), then $\Gamma_b H^P$ and $\Gamma_b H^E$ are bounded away from both infinity and zero for $b = (0, 1)$ and $b = (1, 0)$, provided that (1) and (2) are satisfied.

*Proof.* Clearly we can take $I = \{0, 1\}$. When $|\xi|_I = 1$, $H^P$ and $H^E$ can be explicitly written down:

$$H^E(\eta) = -\frac{\eta_0}{c_{\text{ex}}^+(\eta_0)} + \frac{\eta_1}{c_{\text{ex}}^+(\eta_1)},$$

where $c_{\text{ex}}^+(k) = c_{\text{ex}}(k)$ if $k \neq 0$ and $c_{\text{ex}}^+(0) = 1$; and similarly for $H^P$. So, let $|\xi|_I = 2$. Then $\nabla^-\xi = 0$; hence $H^P = 0$. As for $\nabla^-\eta$, it is first noticed that we may write $H^E(\eta) = \varphi(\eta_0)$ since $E := \eta_0 + \eta_1$ is conserved under the dynamics on $I$. We need to solve the equation

$$c_{\text{zr}}(r)(\varphi(r-1) - \varphi(r)) + c_{\text{zr}}(E-r)(\varphi(r+1) - \varphi(r)) = 2r - E,$$

where $r$ ranges from 1 to $E$. This is uniquely solved up to additive constants. The difference $g(r) := \varphi(r) - \varphi(r+1)$ is recursively determined by

$$g(r) = \frac{E-2r}{c_{\text{zr}}(E-r)} + \frac{c_{\text{zr}}(r)}{c_{\text{zr}}(E-r)}\, g(r-1) \qquad (1 \leqslant r \leqslant E-2)$$

where $g(0)$ may be chosen arbitrarily since $c_{\text{zr}}(1)=0$. We have to show that $g$ is uniformly bounded. By symmetry we observe that $g(r)=g(E-r-1)$, which dispenses the consideration for $r \geqslant E/2$. Recalling that by the hypothesis (2) $c_{\text{zr}}(E-r)-c_{\text{zr}}(r) \geqslant a_2(>0)$ if $2r \leqslant E-k_0$, we set

$$M=\sup_{E\geqslant 1}\ \sup_{r\geqslant 1:2r\leqslant E-k_0} \frac{E-2r}{c_{\text{zr}}(E-r)-c_{\text{zr}}(r)},$$

so that $g(1)\leqslant M$, and

$$\frac{E-2r+c_{\text{zr}}(r)\,M}{c_{\text{zr}}(E-r)} \leqslant M \qquad \text{if} \quad 1\leqslant r\leqslant \frac{E-k_0}{2}.$$

By induction on $r$ we infer that $g(r)\leqslant M$ if $2r\leqslant E-k_0$, and by (1) and (2) $M<\infty$. Hence $g$ is bounded above by a constant independent of $E$. The lower bound may be obtained by replacing the double sup's in the definition of $M$ with double inf's. ∎

In the following discussions the linear space $\mathscr{G}$ (with the natural identification among its elements) will be regarded as a real pre-Hilbert space with the inner product $V=V^{p,\rho}$. By using the reversibility (8) and the identity

$$L_{\{x,x+1\}} \sum_y y(\alpha\xi_y+\beta\eta_y)=\tau_x(\alpha w_{01}^P+\beta w_{01}^E), \tag{11}$$

the following relations (in common with the other models) are easily verified without resorting to Theorem 3 of Subsection 2.3 (see also Lemma 7):

$$V(Lf, g-\tau_1 g)=0; \tag{12}$$

$$V(\alpha w_{01}^P+\beta w_{01}^E, \alpha'\,\nabla^-\xi+\beta'\,\nabla^-\eta)=\underline{\alpha}'\cdot\chi(p,\rho)\,\underline{\alpha}; \tag{13}$$

$$V\{\alpha w_{01}^P+\beta w_{01}^E-\underline{\alpha}\cdot L\underline{f}\}=\underline{\alpha}\cdot\hat{c}(p,\rho;\underline{f})\,\underline{\alpha}. \tag{14}$$

Here $f, g\in\mathscr{F}_c$, $\underline{f}\in\mathscr{F}_c\times\mathscr{F}_c$, and $\underline{\alpha}=(\alpha,\beta)^T$, $\underline{\alpha}'=(\alpha',\beta')^T\in\mathbf{R}\times\mathbf{R}$.

Since $\chi(p,\rho)$ is regular, from (12) and (13) and Theorem 3 we infer that $\{\alpha\,\nabla^-\xi+\beta\,\nabla^-\eta+Lf:\alpha,\beta\in\mathbf{R}, f\in\mathscr{F}_{c,b}\}$ is dense in $\mathscr{G}$. The orthogonal projections of $w_{01}^P$ and $w_{01}^E$ on the two dimensional space spanned by $\nabla^-\xi$ and $\nabla^-\eta$ are of course linear combinations of $\nabla^-\xi$ and $\nabla^-\eta$. Let $D^\circ=D^\circ(p,\rho)$ be the matrix of coefficients in these linear combinations so that the orthogonal projection of $\alpha w_{01}^P + \beta w_{01}^E$ is given by $\underline{\alpha} \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T$. Then

$$\inf_{\underline{f} \in \mathscr{F}_{c,b} \times \mathscr{F}_{c,b}} \sup_{|\underline{\alpha}|=1} V\{\underline{\alpha} \cdot ((w_{01}^P, w_{01}^E)^T - L\underline{f} - D^\circ(\nabla^- \xi, \nabla^- \eta)^T)\} = 0. \tag{15}$$

Let $A = A(p, \rho)$ be a symmetric $2 \times 2$ matrix whose quadratic form is

$$\underline{\alpha} \cdot A \underline{\alpha} = V^{p,\rho}\{\alpha\, \nabla^- \xi + \beta\, \nabla^- \eta\}.$$

**Proposition 5.** The matrix $D^\circ(p, \rho)$ defined as above agrees with $D(p, \rho)$. Moreover it holds that $D(p, \rho)\, A(p, \rho) = \chi(p, \rho)$ and for $\underline{f} \in \mathscr{F}_{c,b} \times \mathscr{F}_{c,b}$

$$\begin{aligned} &V\{\underline{\alpha} \cdot ((w_{01}^P, w_{01}^E)^T - L\underline{f} - D(p, \rho)(\nabla^- \xi, \nabla^- \eta)^T)\} \\ &\quad = \underline{\alpha} \cdot [\hat{c}(p, \rho; \underline{f}) - \hat{c}(p, \rho)]\, \underline{\alpha}. \end{aligned}$$

*Proof.* The proof is an elementary linear algebra in the pre-Hilbert space $(\mathscr{G}, V)$. Put $W = \alpha w_{01}^P + \beta w_{01}^E$. Then from the fact mentioned right before (15) it follows that for every $\underline{\alpha}$ and $\underline{\alpha}' \in \mathbf{R} \times \mathbf{R}$,

$$V(\underline{\alpha}' \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T, W - \underline{\alpha} \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T) = 0.$$

From this together with (13) and (14) we deduce first that

$$V(\underline{\alpha}' \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T, \underline{\alpha} \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T) = \underline{\alpha}' \cdot D^\circ \chi(p, \rho)\, \underline{\alpha},$$

which in particular implies that $D^\circ \chi(p, \rho)$ is symmetric, and then that

$$V\{W - \underline{\alpha} \cdot L\underline{f} - \underline{\alpha} \cdot D^\circ(\nabla^- \xi, \nabla^- \eta)^T\} = \underline{\alpha} \cdot [\hat{c}(p, \rho; \underline{f}) - D^\circ \chi(p, \rho)]\, \underline{\alpha}.$$

Recalling the definition of $D(p, \rho)$ and the formula (15) we now see that $D^\circ = D(p, \rho)$. The identity $D(p, \rho)\, A(p, \rho) = \chi(p, \rho)$ may be seen by replacing $\underline{\alpha}' \cdot D^\circ$ with $\underline{\alpha}' \cdot$ in the second equality above. ∎

**Remark.** Let $\underline{\kappa} = \underline{\kappa}(p, \rho)$ and $\bar{\kappa} = \bar{\kappa}(p, \rho)$ stand for the eigen-values of $D(p, \rho)$ such that $\underline{\kappa} \leqslant \bar{\kappa}$. By employing Lemma 7 in the next subsection we infer that $\underline{\alpha} \cdot A \underline{\alpha} = V\{\alpha\, \nabla^- \xi + \beta\, \nabla^- \eta\} \leqslant E^{\nu_{p,\rho}}[(\Gamma_{01}\{\alpha H^P + \beta H^E\})^2\, c_{01}]$ where $H^P$ and $H^E$ are the functions given in Lemma 4. By using this and the trivial bound $\underline{a} \cdot \hat{c} \underline{a} \leqslant E^{\nu_{p,\rho}}[(\Gamma_{01}\{\alpha \xi_0 + \beta \eta_0\})^2\, c_{01}]$ together with $D = \hat{c} \chi^{-1} = \chi A^{-1}$, we can prove that for some positive constants $m$ and $M$,

$$\frac{m}{p + (1+\alpha)^{-1}} \leqslant \underline{\kappa} \leqslant \bar{\kappa} \leqslant M(1+\alpha) \qquad (\rho \geqslant p > 0),$$

where $\alpha = \alpha(p, \rho)$ is the positive parameter appearing in the definition of $\nu_{p,\rho}$ (ref. 19).

## 2.3. Outline of the Proof of Theorem 3

The proof of Theorem 3 is based on a characterization of a certain class of the closed forms which are associated with the operators $\Gamma_b$. On this topic we shall discuss in Sections 4 and 5. Taking what will be proved in these sections for granted we here give an outline of the proof of Theorem 3 to assure that the proofs for other models can be adapted to the present model. Some of the result given in this section will be applied in succeeding sections. We shall mostly follow the formulation of Varadhan–Yau.[21]

Once the existence of the limiting value $V\{h\}$ is established, the other half of Theorem 3, namely the assertion that $\mathscr{G}_o$ is dense in $\mathscr{G}$ is equivalent that $V\{h\} := V(h, h)$ $(h \in \mathscr{G})$ admits the following variational formula

$$V\{h\} = \sup_{g \in \mathscr{G}_o} [2V(g, h) - V\{g\}].$$

For its proof it suffices to show

$$\limsup_{(m/2n,\, E/2n) \to (p, \rho)} V_{n, m, E}(h, h) \leqslant \sup_{g \in \mathscr{G}_o} [2V(g, h) - V\{g\}], \tag{16}$$

the lower bound being obvious. Moreover if the existence of $V(g, h)$ appearing in the right-hand side is shown (we know of that for $V\{g\}$), this implies the existence of $V\{h\}$, so that the proof of the theorem will be complete.

In the following computation involving $H = L_I^{-1} h$, $I = I(h)$ (cf. Definition 1), the equalities

$$\begin{aligned} \langle u\tau_x h\rangle_{n, m, E} &= -\tfrac{1}{2} \sum_{b \in I^*} \langle \Gamma_{b+x} u \cdot \tau_x (c_b \Gamma_b H)\rangle_{n, m, E} \\ &= -\sum_{b \in I^*} \mathscr{D}^{b+x}_{n, m, E}(u, \tau_x H) \end{aligned} \tag{17}$$

valid for every $\mathscr{F}_{\Lambda(n)}$-measurable function $u$, are useful.

For a local function $f = f(\eta_{-r},\dots, \eta_r)$ and for $\underline{\alpha} = (\alpha, \beta)^T$ we set

$$u^n_{\underline{\alpha}, f} = \sum_{|x| \leqslant n} x(\alpha\xi_x + \beta\eta_x) + \sum_{|x| \leqslant n-r} \tau_x f$$

and

$$\Psi^{y,y+1}_{\underline{\alpha},f} = \Gamma_{y,y+1}(\alpha\xi_y + \beta\eta_y + \tilde{f});$$

also set for $h \in \mathscr{G}$,

$$\Phi^h = \sum_{(x,x+1)\in I^*} \Gamma_{0,1}(\tau_{-x}H) \qquad (H = L^{-1}_{I(h)}h).$$

For $b \in \Lambda^*(K-2r)$ we have $\Gamma_b u^n_{\underline{\alpha},f} = \Psi^b_{\underline{\alpha},f}$. The following lemma is easily shown by using (17).

**Lemma 6.** Let $u^n_{\alpha,f}$, $\Phi^h$, $\Psi^b_{\alpha,f}$ be defined as above. Then

$$\langle u^n_{\underline{\alpha},f}\bar{h}_{0,n'-r}\rangle_{n,m,E} = -\langle \Psi^{0,1}_{\underline{\alpha},f}\Phi^h c_{01}\rangle_{n,m,E}.$$

The next lemma will be applied in various ways. It in particular guarantees that in the definition of $V_{n,m,E}\{h\}$ the contribution of individual $\tau_x h$ is of order $O(1/n)$ uniformly and hence negligible for finding its limit value $V\{h\}$.

**Lemma 7.** Let $h \in \mathscr{G}$ and $a_x$ and $n$ be real constants and a positive integer, respectively. If $\Lambda$ is a finite subset of $\mathbf{Z}$ with $\bigcup_{x\in\Lambda}\tau_x I(h) \subset \Lambda(n)$ and if $F = \sum_{x\in\Lambda} a_x\tau_x h$, then for every $m$ and $E$

$$\langle F(-L_{n,m,E})^{-1} F\rangle_{n,m,E} \leqslant \langle h(-L_I)^{-1} h\rangle_{n,m,E}\ (\sharp I^*) \sum_{x\in\Lambda} |a_x|^2.$$

*Proof.* This proof is the same as given in ref. 21. Let $h = L_I h$. Taking $u = u_n := (-L_{\Lambda(n)})^{-1} F$ (which means that $u_n \in \mathscr{F}_{\Lambda(n)}$ as well as $-L_{\Lambda(n)}u_n = F$) in (17) we see that $\langle F(-L_{\Lambda(n)})^{-1} F\rangle_{n,m,E}$ equals

$$\langle Fu_n\rangle_{n,m,E} = \tfrac{1}{2}\sum_{x\in\Lambda}\sum_{b\in I^*} a_x\langle c_{b+x}\cdot\Gamma_{b+x}\tau_x H\cdot\Gamma_{b+x}u_n\rangle_{n,m,E}.$$

By Schwarz inequality and the assumption on $\Lambda$ we infer that the right hand-side is bounded by $\frac{1}{2}\sqrt{2C\sum_{x\in\Lambda}|a_x|^2}\sqrt{2(\sharp I)\,\mathscr{D}_{n,m,E}\{u_n\}}$, where

$$C := \tfrac{1}{2}\sum_{b\in I^*}\langle(\Gamma_b H)^2 c_b\rangle_{n,m,E} = \langle h(-L_I)^{-1} h\rangle_{n,m,E}.$$

This proves the inequality of the lemma since $\langle Fu_n\rangle_{n,m,E} = \mathscr{D}_{n,m,E}\{u\}$. ∎

We deduce from the identity (11) with the help of Lemma 7 that

$$\langle u^n_{\underline{\alpha},f}\bar{h}_{0,n'}\rangle_{n,m,E} = V_{n,m,E}(\alpha w^P_{01} + \beta w^E_{01} + Lf, h) + O(1/\sqrt{n}),$$

and then from this and Lemma 6 that for $g \in \mathcal{G}_o$, $V(g, h)$ exists and is given by the formula

$$V(\alpha w^P_{01} + \beta w^E_{01} + Lf, h) = E^{\nu_{p,\rho}}[\Psi^{01}_{\underline{\alpha}, f} \Phi^h c_{01}]. \tag{18}$$

On recalling that $\underline{\alpha} \cdot \hat{c}(p, \rho; \underline{f})\, \underline{\alpha} = E^{\nu_{p,\rho}}[(\Psi^{01}_{\underline{\alpha}, f})^2\, c_{01}]$ if $f = \alpha f_1 + \beta f_2$, $\underline{f} = (f_1, f_2)$, this together with (14) yields

$$2V(g, h) - V(g, g) = (2E^{\nu_{p,\rho}}[\Psi^{01}_{\underline{\alpha}, f} \Phi^h c_{01}] - \tfrac{1}{2} E^{\nu_{p,\rho}}[(\Psi^{01}_{\underline{\alpha}, f})^2\, c_{01}]), \tag{19}$$

where $g = \alpha w^P_{01} + \beta w^E_{01} + Lf$. It remains to prove the upper estimate (16) into which the relation (19) is to be substituted.

For each positive constant $C > 0$, let $\mathcal{H}_{p, \rho, C}$ be the set of all functions $\Psi = \Psi(\eta)$ of the form

$$\Psi = \Gamma_{01}(\mathrm{Av}_{|x| \leqslant k}\, \tau_x f), \qquad k \in \mathbf{N}, \quad f \in \mathcal{F}_c$$

with $f$ which satisfies

$$\mathrm{Av}_{|x| \leqslant k}\, E^{\nu_{p,\rho}}[(\Gamma_{01} \tau_x f)^2\, c_{01}] \leqslant C,$$

and define

$$\mathcal{H}_{p,\rho} = \bigcup_{C \geqslant 1} \overline{\mathcal{H}_{p,\rho,C}}^{cv},$$

where $\bar{\mathcal{A}}^{cv}$ denotes the closure of $\mathcal{A} \subset L^2(c_{01} \nu_{p,\rho}, \mathcal{X})$.

**Lemma 8.** For every $h \in \mathcal{G}$,

$$\limsup_{(m/2n, E/2n) \to (p, \rho)} V_{n,m,E}\{h\} \leqslant \sup_{\Psi \in \mathcal{H}_{p,\rho}} (2E^{\nu_{p,\rho}}[\Psi \Phi^h c_{01}] - \tfrac{1}{2} E^{\nu_{p,\rho}}[\Psi^2 c_{01}]).$$

If we show the inclusion

$$\mathcal{H}_{p,\rho} \subset \overline{\{\Psi^{01}_{\underline{\alpha}, f} : \underline{\alpha} \in \mathbf{R} \times \mathbf{R},\, f \in \mathcal{F}_{c,b}\}}^{cv}, \tag{20}$$

then the required upper bound (16) follows from Lemma 8. We postpone the proof of (20) until Sections 4 and 5 since the arguments for it are independent of those given here as well as in the next section where Theorem 1 will be proved. The proof of Lemma 8 is carried out by a compactness argument in $L^2(c_{01} \nu_{p,\rho}, \mathcal{X})$ as in ref. 20 (see also refs. 10 and 21) with the help of the following lemma.

**Lemma 9.** If $h \in \mathcal{G}$ and $u_n = (-L_{\Lambda(n)})^{-1} \sum_{|x|<n'} \tau_x h$, then for each $K > 0$

$$V_{n,m,E}\{h\} = \frac{1}{4n} \sum_{b \in \Lambda^*(n)} \langle (\Gamma_b u_n)^2 c_b \rangle_{n,m,E} \leqslant C_K \qquad \text{if} \quad E/m \leqslant K \tag{21}$$

and

$$V_{n,m,E}\{h\} = \frac{1}{4n} \sum_{|x|<n'} \langle c_{x,x+1} \Gamma_{x,x+1} u_n \cdot \tau_x \Phi^h \rangle_{n,m,E} + o(1),$$

where $o(1)$ vanishes as $n \to \infty$ uniformly in $m$, $E$ as long as $E/m \leqslant K$.

*Proof.* We omit the subscripts $n, m, E$ from $\langle \cdot \rangle_{n,m,E}$. The identity of (21) follows from $V_{n,m,E}\{h\} = \frac{1}{2n} \langle u_n(-L_{\Lambda(n)})\, u_n \rangle$ and the last bound follows from Lemma 7. By the basic relation (17) we observe that

$$\begin{aligned} V_{n,m,E}\{h\} &= -\left\langle u_n \sum_{|x|<n'} \tau_x h \right\rangle \\ &= \frac{-1}{2(2n'-1)} \sum_{|x|<n'} \sum_{b:\, b-x \in I^*} \langle c_b \Gamma_b u_n \cdot \Gamma_b \tau_x H \rangle \\ &= \frac{-1}{2(2n'-1)} \sum_{b \in \mathbf{Z}^*} \sum_{x:\, b-x \in I^*,\, |x|<n'} \langle c_b \Gamma_b u_n \cdot \Gamma_b \tau_x H \rangle. \end{aligned}$$

Noticing that $\sum_{x:\, b-x \in I^*} c_b \Gamma_b \tau_x H = \tau_y(\Phi^h c_{01})$ if $b = (y, y+1)$, from (21) we obtain the bound $|\langle c_b \Gamma_b u_n \cdot \Gamma_b \tau_x H \rangle| \leqslant \sqrt{4nC_K \langle (\Gamma_b H)^2 c_b \rangle}$, so that the contribution of the (unoriented) bonds $\{y, y+1\}$ such that $|y - (\pm n')| \leqslant \sharp I + 1$ is at most $O(1/\sqrt{n})$. Finally substituting the identity $\sum_{x:\, b-x \in I^*} \Gamma_b \tau_x H = \tau_y \Phi^h$ we find the relation of the lemma. ∎

We conclude this section by proving a lemma that supplements Theorem 3: it in particular implies continuous dependence of $D(p, \rho)$ on $(p, \rho)$ in view of the identity $D(p, \rho) = \chi(p, \rho)\, A^{-1}(p, \rho)$ (Proposition 5). If $p = \rho = 0$ and $p = \rho = 1$, we set $V^{0,0} = V^{1,1} = 0$. Notice that if $\rho > p = 1$, the process is reduced to the zero-range process, so that $V^{1,\rho}$ is well defined.

**Lemma 10.** For every $h, g \in \mathcal{G}$, the convergence of $V_{n,m,E}(h, g)$ as the pair $(m/2n, E/2n)$ approaching $(p, \rho)$ is uniform in $p \in [0, 1]$ and $\rho \geqslant p$ as long as the limit is taken under the restriction that $E/m$ is bounded. Moreover $V^{p,\rho}(h, g)$ is jointly continuous with respect to $p \in [0, 1]$ and $\rho \in [p, Kp]$ for each $K$.

*Proof.* It suffices to show that for each $K \geqslant 2$,

$$\lim_{m/2n \to 0} \sup_{E \leqslant Km} V_{n,m,E}\{h\} = 0 \qquad \text{and} \qquad \lim_{(m/2n,\, E/2n) \to (1,1)} V_{n,m,E}\{h\} = 0 \tag{22}$$

since the asserted uniformity of convergence and continuity of the limit function are standard facts resulting from the manner of convergence. If $u = u_n = (-L_{\Lambda(n),m,E})^{-1} \sum_{|x| < n'} \tau_x h$, then by Lemma 7

$$V_{n,m,E}\{h\} \leqslant \tfrac{1}{2} (\sharp I(h)) \sum_{b \in I^*} \langle (\Gamma_b H)^2 c_b \rangle_{n,m,E},$$

of which the right-hand side vanishes in the both limits as desired. ∎

## 3. PROOF OF THEOREM 1

Recall that $P_{\mathrm{eq}} = P_{\mathrm{eq}(p,\rho)}$ is the measure of stationary process on $\mathscr{X}$ starting with $\nu_{p,\rho}$ and a sample path is denoted by $\eta(t)$. For the proof of Theorem 1 we need the following proposition.

**Proposition 11.** Let $h \in \mathscr{G}$ and put $F^N(\eta) = \sqrt{N} \sum_{x \in \mathbf{Z}} J(x/N) \tau_x h(\eta)$. Then, for any $n$ large enough that $h \in \mathscr{F}_{\Lambda(n)}$,

$$E_{\mathrm{eq}}\left[ \sup_{0 \leqslant t \leqslant T} \left| \int_0^t F^N(\eta(N^2 s))\, ds \right|^2 \right] \leqslant \frac{27T}{4} \|J\|^2_{N,L^2} E^\nu[V_{n,|\xi|_{\Lambda(n)},|\eta|_{\Lambda(n)}}\{h\}] + \frac{C_n}{N^2},$$

where $\|J\|^2_{N,L^2} = \frac{1}{N} \sum_{x \in \mathbf{Z}} |J(x/N)|^2$ and $\nu = \nu_{p,\rho}$; the constant $C_n = C_{n,T,h,J}$ may be taken to be $T^2 \|J''\|^2_{L^2} \nu(h^2) n^5$.

*Proof.* The proof is divided into two steps, of which the first step is a special case of Lemmas 4.3 of ref. 5.

*Step 1.* We claim that for every $H \in \mathscr{F}_c$,

$$E_{\mathrm{eq}}\left[ \sup_{0 \leqslant t \leqslant T} \left| \int_0^t (N^2 L H)(\eta(N^2 s))\, ds \right|^2 \right] \leqslant \frac{27}{8} T E^\nu[H(-N^2 L) H]. \tag{23}$$

Analogous inequalities also hold true for localized processes generated by $L_{\Lambda(K)}$. For the proof of (23) we write $y_t$ for $H(\eta(N^2 t))$ and set

$$M_t = y_t - y_0 - \int_0^t (N^2 L H)(\eta(N^2 s))\, ds$$

and

$$M_t^* = y_{T-t} - y_T - \int_{T-t}^{T} (N^2LH)(\eta(N^2s))\, ds$$

so that

$$-\int_0^t (N^2LH)(\eta(N^2s))\, ds = \tfrac{1}{2}\,(M_t + M_T^* - M_{T-t}^*).$$

The process $M_t^*$, $0 \leqslant t \leqslant T$, is a backward martingale, namely it is a martingale relative to the filtration $\mathscr{F}_t^* = \sigma\{Y_{T-s}^N : 0 \leqslant s \leqslant t\}$. Now (23) follows from Doob's inequality and the identities $E_{\mathrm{eq}}\, |M_T|^2 = E_{\mathrm{eq}}\, |M_T^*|^2 = 2TE^{\nu}[H(-N^2L)\, H]$.

*Step 2.* Put $G_n = (2n)^{-1} \sum_{y: |y| < n'} \tau_y h$, where $n'$ is the largest integer among those for which $G_n \in \mathscr{F}_{\Lambda(n)}$. We may then replace $F^N$ by

$$F_n^N := \sqrt{N} \sum_{x \in \mathbf{Z}} J(x/N)\, \tau_x G_n.$$

In fact the error to the expectation which we are to estimate is at most $T^2\, \|J''\|_{L^2}^2\, \nu(h^2)\, n^5$ as is easily computed by noticing that $\nu(h\tau_x h) = 0$ for $|x| > n$ and $|J(u+\delta) + J(u-\delta) - 2J(u)|^2 \leqslant (2\delta^3/3) \int_{-\delta}^{\delta} |J''(u+r)|^2\, dr$.

We may suppose that $J$ vanishes outside a finite interval. Let $(\eta^K(t), P_{\mathrm{eq}}^K)$ be a local process generated by $L_{\Lambda(K)}$ (see the definition of the process $\eta(t)$ given right after Theorem A of Section 1). We apply inequality (23) for $\eta^K(t)$ with $H = (-N^2 L_{\Lambda(K)})^{-1}\, F_n^N$ to see that

$$\begin{aligned} E_{\mathrm{eq}}\left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^t F_n^N(\eta(N^2s))\, ds\right|^2\right] &= \lim_{K \to \infty} E_{\mathrm{eq}}^K \left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^t F_n^N(\eta^K(N^2s))\, ds\right|^2\right] \\ &\leqslant (27/8)\, T \lim_{K \to \infty} E^{\nu}[F_n^N (-N^2 L_{\Lambda(K)})^{-1}\, F_n^N], \end{aligned}$$

where the first equality is due to the fact that if $f \in \mathscr{F}_c$, $\sup_{0 \leqslant t \leqslant T} |\int_0^t f(\eta(s))\, ds|$ is a continuous function on the Skorohod space. To estimate the expectation in the last line we first take conditional expectation given $m = |\xi|_{\Lambda(K)}$ and $E = |\eta|_{\Lambda(K)}$. According to Lemma 7 this conditional expectation is bounded by $(\sharp I^*(G_n))\, \|J\|_{N, L^2}^2\, \langle G_n(-L_{I(G_n)})^{-1}\, G_n \rangle_{K, |\xi|_{\Lambda(K)}, |\eta|_{\Lambda(K)}}$. Since the interval $I(G_n)$ may be chosen to be $\Lambda(n)$, this yields that for $K > n$,

$$E^{\nu}[F_n^N (-N^2 L_K)^{-1}\, F_n^N] \leqslant 4n\, \|J\|_{N, L^2}^2\, E^{\nu}[G_n(-L_{\Lambda(n)})^{-1}\, G_n].$$

On rewriting the right-hand side by means of $V_{n, m, E}$ the inequality of the proposition follows. ∎

**Lemma 12.** Let $v_T^N$ be the largest height of the jumps of $y_t := Y_{t,N}^E(J)$, $0 \leqslant t \leqslant T$, namely $v_T^N = \sup_{0 \leqslant t \leqslant T} |y_t - y_{t-0}|$. Then $\lim_{N\to\infty} P_{\mathrm{eq}}[v_T^N > \varepsilon] = 0$ for every $\varepsilon > 0$.

*Proof.* Since $v_T^N \leqslant N^{-3/2} \sup_{0 \leqslant t \leqslant N^2 T} \sup_x |\nabla J(x/N)| \, [\eta_x(t) + \eta_{x+1}(t)]$, it suffices to show that for each $J \in C_0^\infty(\mathbf{R})$,

$$\lim_{N\to\infty} P_{\mathrm{eq}}\left[ \sup_{0 \leqslant t \leqslant T} \sum_x J(x/N)\, \eta_x(t) > N^{3/2} \right] = 0. \tag{24}$$

By a maximal inequality for reversible Markov processes (cf. ref. 10, p. 346) we have for any measurable $F(\eta)$ and positive number $M$

$$P_{\mathrm{eq}}\left[ \sup_{0 \leqslant t \leqslant N^2 T} |F(\eta(t))| > M \right] \leqslant \frac{e}{M} \sqrt{\nu(F^2) + N^2 T \mathscr{D}\{F\}},$$

where $\mathscr{D}\{F\} = \frac{1}{2} \sum_{b \in \mathbf{Z}} \nu((\Gamma_b F)^2 \, c_b)$. For $F(\eta) = N^{-1} \sum_x J(x/N)\, \eta_x$ we see that $\mathscr{D}\{F\} \leqslant N^{-3} \|\nabla J\|_{N, L^2}^2 \, \nu(\eta_0 c_{01})$ and $\nu(F^2) \leqslant \nu(\eta_0^2) \, \|J\|_{N, L^2}^2$, which clearly imply (24). ▮

*Proof of Theorem 1.* Let $Y_t^N = (Y_{t,N}^P, Y_{t,N}^E)$ be as in Section 1. Let $\underline{J} = (J_1, J_2)^T \in C_0^\infty(\mathbf{R}) \times C_0^\infty(\mathbf{R})$ and for $Y = (Y^P, Y^E)^T \in \mathbf{H}^{(-\lambda)} \times \mathbf{H}^{(-\lambda)}$ write

$$(Y, \underline{J}) = Y^P(J_1) + Y^E(J_2).$$

We are to find a suitable expression of $(Y_t^N, \underline{J})$. To make neat the expression we introduce some notations. Define $Y^N = (Y^{P,N}, Y^{E,N})^T$ by

$$Y^{P,N}(J) = N^{-1/2} \sum_{x \in \mathbf{Z}} J(x/N)(\xi_x - p)$$

and analogously for $Y^{E,N}$; pick up $\underline{f} = (f_1, f_2) \in \mathscr{F}_{c,b} \times \mathscr{F}_{c,b}$ and put

$$F(\eta) = (Y^N, \underline{J}) - \frac{1}{N^{3/2}} \sum_{x \in \mathbf{Z}} \nabla \underline{J}\left(\frac{x}{N}\right) \cdot \tau_x \underline{f}(\eta),$$

where $\nabla J(x/N) = N[J((x+1)/N) - J(x/N)]$. Also put

$$\underline{h}^f(\eta) = (w_{01}^P(\eta), w_{01}^E(\eta))^T - L\underline{f}(\eta) - D(\nabla^- \xi, \nabla^- \eta)^T,$$

$$(R_t^{N,f}, \underline{J}) = \int_0^t \sqrt{N} \sum_x \nabla \underline{J}\left(\frac{x}{N}\right) \cdot \tau_x \underline{h}^f(\eta(N^2 s))\, ds,$$

$$(\delta_t^{N,f}, \underline{J}) = \frac{1}{N^{3/2}} \sum_x \nabla \underline{J}\left(\frac{x}{N}\right) \cdot [\tau_x \underline{f}(\eta(N^2 t)) - \tau_x \underline{f}(\eta(0))].$$

(The underline is omitted from $\underline{f}$ appearing in the super-script.) We compute $N^2 L_{\Lambda(n)} F$, which results in

$$\begin{aligned}(Y_t^N, \underline{J}) - (Y_0^N, \underline{J}) &= F(\eta(N^2 t)) - F(\eta(0)) + (\delta_t^{N,f}, \underline{J}) \\ &= \int_0^t \sqrt{N} \sum_x \nabla \underline{J}\left(\frac{x}{N}\right) \cdot D\tau_x (\nabla^- \xi, \nabla^- \eta)^T (N^2 s)\, ds \\ &\quad + (M_t^{N,f}, \underline{J}) + (R_t^{N,f}, \underline{J}) + (\delta_t^{N,f}, \underline{J}), \end{aligned} \tag{25}$$

where $(\nabla^- \xi, \nabla^- \eta)^T (t) = (\nabla^- \xi(\eta(t)), \nabla^- \eta(\eta(t)))^T$.

$(M_t^{N,f}, \underline{J})$ is a martingale whose quadratic variation process is given by

$$\langle M^{N,f}(\underline{J}) \rangle_t = \int_0^t \Theta_F^N(\eta(N^2 s))\, ds$$

where

$$\Theta_F^N = N^2 \sum_{b \in \mathbf{Z}^*} (\Gamma_b F)^2 c_b. \tag{26}$$

On writing $x_b$ for $\min\{x', x''\}$ if $b = (x', x'')$,

$$\left| \Theta_F^N - \frac{1}{N} \sum_{b \in \mathbf{Z}^*} [\nabla \underline{J}(x_b/N) \cdot (\Gamma_b\{\xi_{x_b} + \tilde{f}_1\}, \Gamma_b\{\eta_{x_b} + \tilde{f}_2\})^T]^2 c_b \right| \leqslant \frac{A_{f,J}^N(\eta)}{N^2},$$

with $\sup_N E_{\mathrm{eq}}[A_{f,J}^N] < \infty$. The law of large numbers therefore yields that as $N \to \infty$,

$$\Theta_F^N(\eta(N^2 t)) \longrightarrow \int_{\mathbf{T}} \underline{J}'(\theta) \cdot \hat{c}(p, \rho; \underline{f})\, \underline{J}'(\theta)\, d\theta,$$

which will identify the variance of the fluctuation term of the limit process. The Centsov's condition for the tightness for $M_t := (M_t^{N,f}, \underline{J})$ is easy to see, e.g.,

$$\begin{aligned} & E_{\mathrm{eq}}[|M_t - M_r|^2 \, |M_r - M_s|] \\ &\quad \leqslant \sqrt{E_{\mathrm{eq}}[(E_{\mathrm{eq}}[|M_t - M_r|^2 \,|\sigma\{M_u : u \leqslant r\}])^2 \, |M_r - M_s|^2]} \\ &\quad \leqslant C_{f,J} \, |t - s|^{3/2} \qquad (0 < s < r < t), \end{aligned}$$

where $C_{f,J}$ may be taken in the form $C_{f,J}^1 + N^{-1/2} C_{f,J}^2$ with $C_{f,J}^1 =$ a constant times $\|\nabla \underline{J}\|_{N,L^2}^3 \sqrt{E^\nu[((|\Gamma_{01}\{\xi_0 - \tilde{f}_1\}|^2 + |\Gamma_{01}\{\eta_0 - \tilde{f}_2\}|^2)\, c_{01})^3]}$. By employing Doob's inequality and noticing $\|\nabla J\|_{N, L^2} \leqslant \|J'\|_{L^2}$ we also observe that

$$E_{\text{eq}}[\sup_{0 \leqslant t \leqslant T} |M_t|^2] \leqslant 8 \|J'\|_{L^2}^2 E^{\nu}\left[(1+\eta_0^2) c_{01} + s_f \sum_b |\Gamma_b \underline{f}|^2 c_b\right], \tag{27}$$

$s_f$ denotes the minimum of $n \geqslant 1$ such that $f \in \mathscr{F}_{\Lambda(n-1)}$.

Clearly

$$\sup_{t \leqslant T} |(\delta_t^{N,f}, \underline{J})| \leqslant 2N^{-1/2} \|\underline{J}'\|_{L^1} \|\underline{f}\|_{\infty}. \tag{28}$$

The term $R_t^{N,f}$ is estimated according to Proposition 11, which combined with Proposition 5 shows that

$$\overline{\lim_{N \to \infty}} E_{\text{eq}}[\sup_{0 \leqslant t \leqslant T} |(R_t^{N,f}, \underline{J})|^2] \leqslant C_T \|\underline{J}'\|_{L^2}^2 \|\hat{c}(p, \rho) - \hat{c}(p, \rho; \underline{f})\|, \tag{29}$$

where $\|A\|$ is the operator norm of $2 \times 2$ matrix relative to the usual inner product in $\mathbf{R} \times \mathbf{R}$.

By summation by parts the first term on the right-hand side of (25) may be written as $\int_0^t Y_s^N(D^T \Delta \underline{J})\, ds$ where $\Delta J(\theta) = N^2[J(\theta + 1/N) + J(\theta - 1/N) - 2J(\theta)]$. Apparently this term would give the drift term of the equation in the limit. It is convenient to consider the process

$$(\tilde{Y}_t^N, \underline{J}) := (Y_t^N, \underline{J}) - (R_t^{N,f}, \underline{J}),$$

in terms of which Eq. (25) may be rewritten as

$$(\tilde{Y}_t^N, \underline{J}) = (\tilde{Y}_0^N, \underline{J}) + \int_0^t (\tilde{Y}_s^N, D^T \Delta \underline{J})\, ds + \int_0^t (R_s^{N,f}, D^T \Delta \underline{J})\, ds$$
$$+ (M_t^{N,f}, \underline{J}) + (\delta_t^{N,f}, \underline{J}).$$

We have the bound $E_{\text{eq}} |(Y_s^N, D^T \Delta \underline{J})|^2 \leqslant C \|\underline{J}''\|_{L^2}^2$ (with $C = 2[p(1-p) + \nu((\eta_0 - \rho)^2)] \|D\|$), from which we infer that the laws of the processes

$$\int_0^t (\tilde{Y}_s^N, D^T \Delta \underline{J})\, ds + \int_0^t (R_s^{N,f}, D^T \Delta \underline{J})\, ds = \int_0^t (Y_s^N, D^T \Delta \underline{J})\, ds$$

constitute a tight family. Hence, by virtue of the equation above, the same is true also for the processes $(\tilde{Y}_t^N, \underline{J})$.

Let $Y_t^f$, $A_t^f$, and $M_t^f$ be limit processes of $\tilde{Y}_t^N$, $\int_0^t R_s^{N,f}\,ds$, and $M_t^{N,f}$, respectively. By virtue of Lemma 12 these are all continuous processes. We have the following stochastic equation

$$(Y_t^f, \underline{J}) = \int_0^t (Y_s^f, D^T \underline{J}'')\,ds + (A_t^f, D^T \underline{J}'') + (M_t^f, \underline{J}).$$

The sum of the first two terms on the right side is to be a limit process of $\int_0^t (Y_s^N, D^T \underline{J}'')\,ds$, hence its law is independent of $f$. The third term is a continuous martingale whose quadratic variation process is $(\underline{J}', \hat{c}(p, \rho; f)\,\underline{J}')_{L^2}\,t$; hence, by Lévy's theorem, it is a Brownian motion. It in particular follows that the family of laws of processes $(M^f, \underline{J})$ is tight if $f$ is varied so that $\|\hat{c}(p,\rho) - \hat{c}(p,\rho; f)\| \to 0$, in which case the tightness for processes $(Y_t^f, \underline{J})$ also follows and, in view of (29), any limit process, $(Y_t, \underline{J})$ say, satisfies

$$(Y_t, \underline{J}) = \int_0^t (Y_s, D^T \underline{J}'')\,ds + (B_t, \underline{J}), \tag{30}$$

where $(B_t, \underline{J})$ is a Brownian motion whose variance is $(\underline{J}', \hat{c}(p,\rho)\,\underline{J}')_{L^2}\,t$.

We claim that a sequence of $f_N$ can be suitably chosen so that the sequence of laws of processes $(\tilde{Y}_t^N, \underline{J})$, $0 \leqslant t \leqslant T$, $N = 1, 2,\ldots$ is tight for every $\underline{J}$ and every limit process is a solution of the stochastic integral equation (30). For the proof first observe, with the help of (27), that $M^{N,f}$, as an $\mathbf{H}^{(-\lambda)}$-valued process, converges in law to $M^f$, which, being an $\mathbf{H}^{(-\lambda)}$-valued Brownian motion, in turn converges to the Brownian motion $B$ as $\hat{c}(p,\rho; f) \to \hat{c}(p,\rho)$. This shows that we can choose $f_N$ so that the family of the laws of $\mathbf{H}^{(-\lambda)}$-valued processes $M^{N,f_N}$ is tight and $\hat{c}(p,\rho; f_N) \to \hat{c}(p,\rho)$. We can modify this choice of $f_N$ so that it in addition holds true that $N^{-1/2}\,\|f_N\|_\infty \to 0$, so that owing to (28)

$$E_{\mathrm{eq}}\Big[\sup_{0 \leqslant t \leqslant T} \big[|(\delta_t^{N,f_N}, \underline{J}))| + |(R_t^{N,f_N}, \underline{J}))|^2\big]\Big] \to 0 \qquad (N \to \infty)$$

for every $J \in C_0^\infty(R)$. This proves the claim, in particular the tightness for $(Y_t^N - R_t^{N,f_N}, \underline{J})$. The tightness for $(Y_t^N, \underline{J})$ incidentally follows and any limit process of it solves (30). Finally the tightness of $Y_t^N$ as an $\mathbf{H}^{(-\lambda)}$-valued process (for $\lambda > 2$) follows from Lemma 13 below. Since the solution of (30) is unique in law, the proof of Theorem 1 is now complete. ▌

**Lemma 13.** For $\underline{J} = (J_1, J_2)^T \in C_0^\infty(\mathbf{R}) \times C_0^\infty(\mathbf{R})$,

$$E_{\mathrm{eq}}\Big[\sup_{0 \leqslant t \leqslant T} |(Y_t^N, \underline{J})|^2\Big] \leqslant C_1 T\,\|\underline{J}'\|_{L^2}^2.$$

*Proof.* The lemma follows from inequality (23) applied with $H(\eta) = (Y^N, \underline{J})(\eta)$ since $2E^\nu[H(-N^2L)\,H] = E^\nu[\Theta_H^N] \leqslant C\,\|\underline{J}'\|_{N,L^2}^2$ where $\Theta_F^N$ is defined by (26) (for the last inequality set $f = 0$ in (27)). ▮

## 4. CLOSED FORM

Fixing the parameters $p \in (0, 1)$ and $\rho > p$, we denote by $\mathbf{E}$ the expectation under the measure $\nu = \nu_{p,\rho}$. This and the next sections will not concern the process measure $P_{\mathrm{eq}}$ at all.

Let $\Lambda$ be an interval of $\mathbf{Z}$. Suppose we are given a function $G$ of $\eta \in \mathbf{Z}_+^\Lambda$ and define its "gradient" along the oriented bond $b \in \Lambda^*$ by

$$\Psi^b(\eta) := \Gamma_b G(\eta) = G(S^b\eta) - G(\eta). \tag{31}$$

Then $\Psi^b$ can be "integrated" to recover $G$ up to an additive constant by the formula

$$\sum_{k=1}^{n} \Psi^{b(k)}(\eta(k-1)) = G(\eta(n)) - G(\eta(0)) \tag{32}$$

which holds for every $S$-chain $\{\eta(k)\}_{k=0}^n$ on $\Lambda$, namely for every sequence $\eta(k)$, $k = 0,\ldots, n$ in $\mathbf{Z}_+^\Lambda$ such that for every $k$,

$$\exists b(k) \in \Lambda^*, \qquad \eta(k) = S^{b(k)}\eta(k-1).$$

Conversely suppose we are given a set of functions $\Psi^b$ on $\mathbf{Z}_+^\Lambda$ such that the sum on the left-hand side of (32) vanishes whenever the chain $\{\eta(k)\}_{k=0}^n$ is closed in the sense that $\eta(n) = \eta(0)$. Then the formula (32) defines a function $G$ (on $\mathbf{Z}_+^\Lambda$) up to an additive constant and $\Psi^b$ is the "gradient" of $G$. We call a set of functions $\Psi^b$ described above a *closed form* on $\Lambda$. A family of functions $\Psi^b$ on $\mathscr{X}$ is said to be closed (on $\mathbf{Z}$), if $\Lambda$ is a finite interval of $\mathbf{Z}$ and $\Psi^b$ ($b \in \Lambda^*$) regarded as functions on $\mathbf{Z}_+^\Lambda$ with each configuration outside $\Lambda$ frozen constitutes a closed form on $\Lambda$. From (31) it follows that

$$\Psi^{y,x}(\eta) = G(S^{y,x}\eta) - G(\eta) = -G(S^{x,y}S^{y,x}\eta) + G(S^{y,x}\eta) = -\Psi^{x,y}(S^{y,x}\eta),$$

if $S^{y,x}\eta \neq \eta$ and $\Psi^{y,x}(\eta) = 0$ if $S^{y,x}\eta = \eta$. Thus if $b = (x, y) \in \Lambda^*$ and $b' = (y, x)$, then

$$\Psi^{b'}(\eta) = -\Psi^b(S^{b'}\eta)\,\mathbf{1}(S^{b'}\eta \neq \eta).$$

**Definition 2.** A function $\Psi \in L^2(c_{01}\nu) = L^2(c_{01}\nu_{p,\rho}, \mathscr{X})$ is called a germ of a translation covariant closed form or simply a germ, if the family $\{\Psi^b\}$ given by the following two relations

(i) $\Psi^{x,x+1} = \tau_x \Psi$;

(ii) $\Psi^{x+1,x}(\eta) = -\Psi^{x,x+1}(S^{x+1,x}\eta)\, \mathbf{1}(S^{x+1,x}\eta \neq \eta)$

constitutes a closed form on $\mathbf{Z}$.

Every element of $\{\Gamma_{01}\tilde{f}: f \in \mathscr{F}_c\}$ is a germ. Both

$$\Gamma_{01}\{\xi_0\} = -(1-\xi_1)\,\xi_0 \qquad \text{and} \qquad \Gamma_{01}\{\eta_0\} = -(1-\xi_1)\,\eta_0 - \mathbf{1}(\eta_0 \geqslant 2)\,\xi_1$$

are also germs since $\tau_x \Gamma_{01}\{\eta_0\} = \Gamma_{x,x+1}\{\eta_x\} = -\Gamma_{x,x+1}\{\sum_y y\eta_y\}$ and similarly for $\Gamma_{01}\{\xi_0\}$. The next theorem states that every germ is a limit of the linear combinations of these functions, namely functions of the form

$$\Psi^{01}_{\alpha,\beta,f} := \Gamma_{01}\{\alpha\xi_0 + \beta\eta_0 + \tilde{f}\}.$$

**Theorem 14.** The set of all germs of translation covariant closed forms agrees with the closure of the space $\{\Psi^{01}_{\alpha,\beta,f}: \alpha, \beta \in \mathbf{R}, f \in \mathscr{F}_c\}$ in $L^2(c_{01}\nu)$.

*Proof of The Inclusion Relation (20).* By virtue of Theorem 14 it suffices to prove that every element of $\mathscr{H}_{p,\rho}$ is a germ. Let $\Psi \in \mathscr{H}_{p,\rho}$. According to the definition of $\mathscr{H}_{p,\rho}$ there exists a sequence of functions $f_K \in \mathscr{F}_c$ $(K \in \mathbf{N})$ such that $\mathrm{Av}_{|x| \leqslant K}\, \mathbf{E}[(\Gamma_{01}\tau_x f_K)^2\, c_{01}] \leqslant C$ and if $\Psi_K := \Gamma_{01}\, \mathrm{Av}_{|x| \leqslant K}\, \tau_x f_K$, then the sequence $(\Psi_K)$ converges to $\Psi$ strongly in $L^2(c_{01}\nu)$. From the condition on $f_K$ just mentioned it is clear that $\sup_{|x| \leqslant K} \mathbf{E}[(\Gamma_{01}\tau_x f_K)^2\, c_{01}] \leqslant (2K+1)\, C$; in particular

$$\lim_{K\to\infty} K^{-2} \sup_{|x| \leqslant K} \mathbf{E}[(\Gamma_{01}\tau_x f_K)^2\, c_{01}] = 0. \tag{33}$$

Set $\Psi^{x,x+1}_K := \tau_x \Psi_K$. Then it is not hard to deduce from (33) that any limit of $(\Psi^b_K)_b$ is a closed form. Consequently, $\Psi = \lim \Psi_K$ is a germ. ∎

The rest of this section (together with the next one) is devoted to the proof of Theorem 14. We shall adapt the lines of refs. 8 and 11.

We introduce the truncated conditional expectation

$$\Psi^b_n := \mathbf{E}[\Psi^b \mid \mathscr{F}_{\Lambda(2n)}] \cdot \mathbf{1}\left(\frac{|\xi|_{\Lambda(2n)}}{4n+1} > \frac{p}{2},\ \frac{|\eta|_{\Lambda(2n)}}{4n+1} < 2\rho\right).$$

Since the transformations $S^b$, $b \in \Lambda(2n)$ commute with the conditional expectation and both $|\xi|_{\Lambda(2n)}$ and $|\eta|_{\Lambda(2n)}$ are invariant under them, the set of functions $\Psi_n^b$ constitutes a closed form on $\Lambda(2n)$. Hence for each $n$ we can find a function $G_{2n} \in \mathscr{F}_{\Lambda(2n)}$ such that

$$\Gamma_b G_{2n}(\eta) = \Psi_n^b \qquad \text{for} \quad b \in \Lambda^*(2n),$$
$$\langle G_{2n} \rangle_{2n, m, E} = 0 \qquad \text{for} \quad m = 1, 2, \ldots, \quad \text{and} \quad E \geqslant m$$

and

$$G_{2n}(\eta) = 0 \qquad \text{unless} \quad \frac{|\xi|_{\Lambda(2n)}}{4n+1} > \frac{p}{2}, \quad \frac{|\eta|_{\Lambda(2n)}}{4n+1} < 2\rho.$$

Put

$$f_n = \frac{1}{2n} \mathbf{E}[G_{2n} \mid \mathscr{F}_{\Lambda(n)}],$$

and

$$\Psi_n = \sum_y \Gamma_{01} \tau_y f_n = \Gamma_{01} \tilde{f}_n.$$

We are to show that the functions $\Psi_n$ converge along a subsequence, weakly in $L^2(c_{01} \nu)$, to a function of the form $\Psi + \Gamma_{01}\{\alpha\xi_0 + \beta\eta_0\}$. This will show that the set of germs is included in the closure of the set of $\Psi^{01}_{\alpha, \beta, f}$, and thus conclude the proof since the inclusion in the opposite direction is obvious.

Define

$$s_n^+ = \tau_{-n} \Gamma_{n, n+1} f_n \qquad \text{and} \qquad s_n^- = \tau_{n+1} \Gamma_{-n-1, -n} f_n.$$

Then, on using $\tau_y \Gamma_{x, z} f = \Gamma_{x+y, z+y} \tau_y f$, $\Psi_n$ may be written as

$$\Psi_n = \Gamma_{01} \sum_{(-y, -y+1) \in \Lambda^*(n)} \tau_y f_n + s_n^+ + s_n^-.$$

The first term equals $(2n)^{-1} \sum_{(-y, -y+1) \in \Lambda^*(n)} \tau_y \Psi_n^{-y, -y+1}$, which converges to $\Psi$ as $n \to \infty$ strongly in $L^2(c_{01} \nu)$.

**Lemma 15.** The expectations $\mathbf{E}[|s_n^{\pm}|^2 c_{01}]$ are bounded.

Taking this lemma for granted for the time being we prove that if $s^{\pm}$ is a weak limit point of $s_n^{\pm}$ in $L^2(c_{01}\nu)$, then they are necessarily of the form

$$s^{\pm} = \Gamma_{01}\{\alpha^{\pm}\xi_0 + \beta^{\pm}\eta_0\} \tag{34}$$

with some constants $\alpha^{\pm}$ and $\beta^{\pm}$. This implies the assertion of Theorem 14 since the strong closure and the weak closure of a convex set in a Hilbert space coincides. In what follows we shall provide a proof of (34), in which, although we can follow ref. 11 (see also refs. 10 and 14), we shall proceed somewhat differently.

First consider $s^+$. We introduce the operators $\pi_x^k$ and $\pi_x^-$ which are defined by

$$\pi_x^k f(\eta) = f(R_x^k\eta) - f(\eta) \qquad (k = 0, 1, 2,...)$$

and

$$\pi_x^- f(\eta) = f(\eta - \delta_x) - f(\eta) \qquad (\eta_x \geqslant 2),$$

where $R_x^k\eta$ (resp. $\eta - \delta_x$) stands for the configurations which are obtained from $\eta$ by replacing its (spin) value at the site $x$ by $k$ (resp. by reducing its (energy) value at $x$ by 1). By means of these operators $s_n^+$ may be written in the form

$$s_n^+ = \Gamma_{01} h_n = \xi_0(1 - \xi_1)\, \pi_0^0 h_n + \mathbf{1}(\eta_0 \geqslant 2)\, \xi_1 \pi_0^- h_n \tag{35}$$

where

$$h_n = \tau_{-n} f_n = \frac{1}{2n} \mathbf{E}[\tau_{-n} G_{2n} \mid \mathscr{F}_{\{-2n,..., -1, 0\}}]. \tag{36}$$

It would be clear that $s^+$ is a function of $\{\eta_x : x \leqslant 1\}$. Since $\Gamma_b G_{2n}$ is bounded in $L^2(\nu)$ if $b \in (\Lambda(2n))^*$ and $\Gamma_b$ commutes with the operation of taking weak limit, it holds that

$$\Gamma_{x-1,x} s^+ = 0 \qquad \text{for} \quad x < 0.$$

In view of Hewitt–Savage zero-one law, these imply that $s^+$ depends only on $\{\eta_0, \eta_1\}$. We then infer from (35) and (36) that it is of the form

$$s^+(\eta) = \xi_0(1 - \xi_1)\, \varphi(\eta_0) + \mathbf{1}(\eta_0 \geqslant 2)\, \xi_1 \psi(\eta_0)$$

where $\varphi$ and $\psi$ are functions on $\mathbf{N}$ and $\{2, 3,...\}$, respectively. Incidentally, we have, as $n \to \infty$ (along an appropriate subsequence),

$$\xi_0 \pi_0^0 h_n(\eta) \to \xi_0 \varphi(\eta_0) \qquad \text{and} \qquad \mathbf{1}(\eta_0 \geqslant 2)\, \pi_0^- h_n(\eta) \to \mathbf{1}(\eta_0 \geqslant 2)\, \psi(\eta_0), \tag{37}$$

where the convergence is weak in $L^2(\nu, \mathscr{X})$. Notice that (37) is valid irrespectively of the values of $\xi_1$ since $h_n$ does not depend on it.

Our task is to prove that $\psi$ is constant and

$$\varphi(k) = \psi(2)\, k + \text{const}$$

since $\Gamma_{01}\{\alpha\xi_0 + \beta\eta_0\} = \xi_0(1-\xi_1)(-\beta\eta_0 - \alpha) - \beta\mathbf{1}(\eta_0 \geqslant 2)\, \xi_1$. For the proof we consider the identity

$$\pi_{0,-1}^{\text{zr}} s_n^+(\eta) = s_n^+(S_{\text{zr}}^{0,-1}\eta) - s_n^+(\eta)$$

when $\eta_0 \geqslant 3$, $\xi_{-1} = 1$. The first term on the right side converges to

$$s^+(S_{\text{zr}}^{0,-1}\eta) = s^+(\eta - \delta_0).$$

We compute the limit of the left hand side when $\xi_1 = 1$. Since $\xi_1 s_n^+ = \xi_1 \mathbf{1}(\eta_0 \geqslant 2)\, \pi_0^- h_n$ and since $\pi_0^-$ and $\pi_{0,-1}^{\text{zr}}$ commute if $\eta_0 \geqslant 3$, $\xi_1 = \xi_{-1} = 1$, we deduce that for such configurations,

$$\pi_{0,-1}^{\text{zr}} s_n^+ = \pi_{0,-1}^{\text{zr}} \pi_0^- h_n = \frac{1}{2n}\, \xi_{-1} \pi_0^- \,(\tau_{-n} \Psi_n^{n-1,n})$$

which vanishes in the weak limit. As a net result we have

$$s^+(\eta) = s^+(\eta - \delta_0) \qquad \text{if} \quad \eta_0 \geqslant 3, \quad \xi_1 = 1.$$

This shows that $\psi(k) = \psi(k-1)$ for $k \geqslant 3$, hence $\psi$ is constant.

The rest is easy. Indeed expressing $\pi_0^0 h_n$ by means of $\pi_0^- h_n$ as follows

$$\begin{aligned}(1-\xi_1)\, s_n^+(\eta) &= (1-\xi_1)\, \xi_0 \pi_0^0 h_n(\eta) \\ &= \xi_0(1-\xi_1)[\pi_0^- h_n(\eta) + \pi_0^- h_n(\eta - \delta_0) \\ &\quad + \cdots + \pi_0^- h_n(\eta - (\eta_0 - 2)\, \delta_0) + \pi_0^0 h_n(\eta - (\eta_0 - 1)\, \delta_0)]\end{aligned}$$

$(\eta_0 \geqslant 2)$ and passing to the limit with the help of (37) we conclude that

$$(1-\xi_1)\, s^+ = \xi_0(1-\xi_1)[(\eta_0 - 1)\, \psi(2) + \varphi(1)].$$

Hence $\varphi(k) = \psi(2)\, k + \varphi(1) - \psi(2)$ as required.

As for $s^-$ we have

$$s_n^- = \Gamma_{01} h_n^- = \xi_0(1-\xi_1)\,\pi_1^{\eta_0} h_n^- + \mathbf{1}(\eta_0 \geqslant 2)\,\xi_1 \pi_1^+ h_n^-$$

where $h_n^- = \tau_{n+1} f_n = (2n)^{-1}\,\mathbf{E}[\tau_{n+1} G_{2n} \mid \mathscr{F}_{\{1,2,\ldots,2n+1\}}]$ and $\pi_x^+ f = f(\eta+\delta_x) - f(\eta)$, and derive, as before,

$$s^-(\eta) = \xi_0(1-\xi_1)\,\varphi(\eta_0) + \mathbf{1}(\eta_0 \geqslant 2)\,\xi_1 \psi(\eta_1).$$

This time we have $\xi_2 \pi_{1,2}^{\mathrm{zr}}$ act on $s_n^-$ on the set $\{\eta_1 \geqslant 2\}$ to prove that $\psi$ is constant. In the same way as in the case of $s^+$ we accordingly deduce that $\varphi(k) = \psi(1)\,k + \varphi(1) - \psi(1)$.

Theorem 3 has been proved by taking Lemma 14 for granted. The proof of Lemma 14, being involved, will be given in the next section.

## 5. PROOF OF LEMMA 15

We prove the boundedness of $\mathbf{E}[|s_n^+|^2\, c_{01}]$ only. The proof for $\mathbf{E}[|s_n^-|^2\, c_{01}]$ is the same. The parameters $p$ and $\rho$ are fixed and often omitted from the notations. We recall that $s_n^+ = \Gamma_{01} h_n$, with $h_n = \frac{1}{2n}\,\mathbf{E}[\tau_{-n} G_{2n} \mid \mathscr{F}_{\{-2n,\ldots,-1,0\}}]$, where $\mathbf{E}$ indicates the expectation by $\nu$. For the proof we need the following properties of $G_{2n} \in \mathscr{F}_{\Lambda(2n)}$ (and only these).

(a) $\sup_{b \in \Lambda^*(2n)} \mathscr{D}_{p,\rho}^b\{G_{2n}\} \leqslant C$;

(b) $\langle G_{2n} \rangle_{2n,m,E} = 0$ for $m \geqslant 1$, $E \geqslant m$;

(c) $G_{2n}(\eta) = 0$ unless $\frac{1}{|\Lambda(2n)|}\,|\xi|_{\Lambda(2n)} > \frac{p}{2}$, $\frac{1}{|\Lambda(2n)|}\,|\eta|_{\Lambda(2n)} < 2\rho$.

where $\mathscr{D}_{p,\rho}^b\{G_{2n}\} := \frac{1}{2}\,\mathbf{E}[|\Gamma_b G_{2n}|^2\, c_b]$. In order to deduce the required bound from these conditions the following result from ref. 15 is fundamental.

**Theorem B.** Suppose that the conditions (1) through (3) are satisfied. Then there exists a constant $C$ such that for all positive integers $n$, $m$, and $E$, satisfying $m \leqslant |\Lambda(n)|$ and $E \geqslant m$, and for all real functions $f$ on $\mathbf{Z}_+^{\Lambda(n)}$,

$$\langle (f - \langle f \rangle_{n,m,E})^2 \rangle_{n,m,E} \leqslant C\,\frac{E}{m} \cdot n^2 \mathscr{D}_{n,m,E}(f). \tag{38}$$

The next lemma also is a consequence of a result of ref. 15. The definition of transformation $S_{\mathrm{zr}}^{x,y}$ (and accordingly of the operator $\pi_{x,y}^{\mathrm{zr}}$) is extended to all pairs $x, y$ in $\mathbf{Z}$ that are not necessarily adjacent to each other.

**Lemma 16.** Suppose the condition (3) to hold. Then there exists a constant $C$ such that for all real functions $f$ on $\mathbf{Z}_+^{\Lambda(n)}$ and for $x \in \Lambda(n)$,

$$\sum_{y:\, y \neq x} \langle c_{\mathrm{zr}}(\eta_x)\, \xi_y (\pi^{\mathrm{zr}}_{x,y} f)^2 \rangle_{n,m,E} \leqslant C n^2 E \cdot \sup_{b \in \Lambda^*(n)} \mathscr{D}^b_{n,m,E}\{f\}.$$

*Proof.* In ref. 15 (Lemma 4) it is shown (under the condition (3)) that for any $x, y \in \Lambda(n)$ $(x \neq y)$, the expectation $\langle c_{\mathrm{zr}}(\eta_x)\, \xi_y [\pi^{\mathrm{zr}}_{x,y} f]^2 \rangle_{n,m,E}$ is bounded by a constant multiple of

$$|x-y| \sum_{b \in (I[x,y])^*} \langle (\Gamma_b f)^2\, c_b \xi_y + (\pi^{\mathrm{ex}}_b f)^2\, c_{\mathrm{zr}}(\eta_y) \rangle_{n,m,E},$$

where $I[x, y]$ stands for the interval whose end points are $x$ and $y$. Taking summation over $y \neq x$ and dominating $|x-y|$ by $2n$, $\sum_y c_{\mathrm{zr}}(\eta_y)$ by $a_1 E$, $m$ by $E$ and $(\pi^{\mathrm{ex}}_b f)^2$ by $(\Gamma_b f)^2\, c_b / a_0$ we find the inequality of the lemma to hold. ∎

Now we proceed into the proof of Lemma 15. Let $\nu^1 = \nu^1_{p,\rho}$ denote the one site marginal of $\nu_{p,\rho}$:

$$\nu^1(r) = \nu_{p,\rho}(\{\eta\colon \eta_0 = r\}),$$

and $H_r$, $r = 0, 1, 2,\ldots$, the functions of $\zeta_n := (\eta_{-2n},\ldots, \eta_{-1})$ defined by

$$H_r(\zeta_n) = h_n|_{\eta_0 = r} = \frac{1}{2n}\, \mathbf{E}[\tau_{-n} G_{2n} \mid \eta_0 = r, \mathscr{F}_{\{-2n,\ldots,-1\}}]$$

(where $\mathbf{E}[\,\cdot \mid \eta_0 = r, \mathscr{F}] = \mathbf{E}[\,\cdot \cap \{\eta_0 = r\} \mid \mathscr{F}] / \mathbf{P}[\eta_0 = r]$). Then

$$\begin{aligned}
\mathbf{E}[|s_n^+|^2\, c_{01}] &= \mathbf{E}[|\Gamma_{01} h_n|^2\, c_{01}] \\
&= \mathbf{E}[|\pi^{\mathrm{ex}}_{01} h_n|^2\, c_{\mathrm{ex}}(\eta_0)(1-\xi_1)] + \mathbf{E}[|\pi^{\mathrm{zr}}_{01} h_n|^2\, c_{\mathrm{zr}}(\eta_0)\, \xi_1] \\
&= (1-p) \sum_{r=1}^{\infty} \nu^1(r)\, c_{\mathrm{ex}}(r)\, \mathbf{E}\, |H_r - H_0|^2 \\
&\quad + p\alpha \sum_{r=1}^{\infty} \nu^1(r)\, \mathbf{E}\, |H_{r+1} - H_r|^2. \qquad (39)
\end{aligned}$$

Here $\alpha = \alpha(p, \rho)$ and in the last equality we applied the relation that $\nu^1(r)\, c_{\mathrm{zr}}(r) = \alpha \nu^1(r-1)$.

In the first two lemmas (but not in the third) that follow we fix $\zeta_n$ and treat $H_r$ as a function of $r$. The estimates concerning $H_r$ given below will be uniform in $\zeta_n$. The expectation by the distribution $\nu^1$ on the variable $\eta_0$ is denoted by $\mathbf{E}_0$.

**Lemma 17.** There exists a constant $C$ independent of $n$ and $\zeta_n$ such that

$$p\mathbf{E}_0[(H_{\eta_0}-\mathbf{E}_0[H_{\eta_0}\mid\xi_0=1])^2\mid\xi_0=1]\leqslant C\sum_{r=1}^{\infty}v^1(r)\,|H_{r+1}-H_r|^2.$$

*Proof.* This is nothing but Lemma 4.2 (one site spectral gap estimate) of ref. 12, where it is shown that $C$ can be independent of $\rho$ (for the present purpose it may depend on $\rho$). ∎

**Lemma 18.** There exists a constant $C$ independent of $n$ and $\zeta_n$ such that

$$\sum_{r=1}^{\infty}v^1(r)\,c_{\text{ex}}(r)\,|H_r-H_0|^2$$

$$\leqslant C\sum_{r=1}^{\infty}v^1(r)\,|H_{r+1}-H_r|^2+C(\mathbf{E}_0[H_{\eta_0}\mid\xi_0=1]-H_0)^2.$$

*Proof.* On using $c_{\text{zr}}(r)\,v^1(r)=\alpha v^1(r-1)$, $r\geqslant 2$, the left-hand side of the inequality of the lemma is bounded by

$$v^1(1)\,c_{\text{ex}}(1)\,|H_1-H_0|^2+2\bar{a}\sum_{r=2}^{\infty}\alpha v^1(r-1)\,|H_{r-1}-H_0|^2$$

$$+2\bar{a}\sum_{r=2}^{\infty}\alpha v^1(r-1)\,|H_r-H_{r-1}|^2,$$

where $\bar{a}=\sup_{r\geqslant 2}c_{\text{ex}}(r)/c_{\text{zr}}(r)$. Setting $\hat{H}_1=\mathbf{E}_0[H_{\eta_0}\mid\xi_0=1]$ we bound the sum of the first two terms by a constant multiple of

$$\sum_{r=1}^{\infty}v^1(r)\,|H_r-H_0|^2\leqslant\sum_{r=1}^{\infty}v^1(r)[2\,|H_r-\hat{H}_1|^2+2\,|\hat{H}_1-H_0|^2].$$

Now the required inequality follows from the preceding lemma. ∎

**Lemma 19.** For some constant $C$, $\sum_{r=1}^{\infty}v^1(r)\,\mathbf{E}\,|H_{r+1}-H_r|^2\leqslant C$.

*Proof.* Fixing $\zeta_n=(\eta_{-2n},\dots,\eta_{-1})$ (and $n$) as before, we consider a function $f=f_{\zeta_n}(\eta_0,\dots,\eta_n)$ defined by

$$f(\eta_0,\eta_1,\dots,\eta_n)=\frac{1}{2n}\mathbf{E}[\tau_{-n}G_{2n}\mid\zeta_n,\eta_0,\eta_1,\dots,\eta_n].$$

Let $\mathbf{P}_n$ denote the probability law of the variables $\eta_0, \eta_1,..., \eta_n$ under $\mathbf{P} = \nu_{p,\rho}$ and $\mathbf{E}_n$ the expectation by $\mathbf{P}_n$. Then

$$\begin{aligned} H_{r+1} &= \mathbf{E}_n[f \mid \eta_0 = r+1] \\ &= \sum_{x=1}^{n} \mathbf{E}_n[Y^{-1}\xi_x f; Y > 0 \mid \eta_0 = r+1] + R_{r+1} \end{aligned}$$

where $R_{r+1} = \mathbf{E}_n[f; Y = 0 \mid \eta_0 = r+1]$ and

$$Y = \xi_1 + \cdots + \xi_n.$$

By the identity $\nu^1(r+1)\, c_{\mathrm{zr}}(r+1) = \nu^1(r)\, \alpha$ and the reversibility we see that

$$\mathbf{E}_n[\xi_x f \mid \eta_0 = r+1, \sigma\{Y\}] = \frac{1}{\alpha} \mathbf{E}_n[c_{\mathrm{zr}}(\eta_x)(f \circ S_{\mathrm{zr}}^{x,0}) \mid \eta_0 = r, \sigma\{Y\}],$$

provided that $r \geqslant 1$. ($\sigma\{Y\}$ denotes the $\sigma$-field generated by $Y$.) Define $M = M(\eta_1,..., \eta_n)$ by

$$M = \begin{cases} (\alpha Y)^{-1} \sum_{x=1}^{n} c_{\mathrm{zr}}(\eta_x) & \text{if } \; Y \neq 0 \\ 1 & \text{if } \; Y = 0. \end{cases}$$

By conditioning on $\xi_1,..., \xi_n$ it is assured that $\mathbf{E}_n[M \mid \eta_0] = 1$. From these identities we deduce that

$$\begin{aligned} H_{r+1} - H_r &= \sum_{x=1}^{n} \mathbf{E}_n[(\alpha Y)^{-1}\, c_{\mathrm{zr}}(\eta_x)\, \pi_{x,0}^{\mathrm{zr}} f; Y \geqslant 1 \mid \eta_0 = r] \\ &\quad + \mathbf{E}_n[(M-1)\, f \mid \eta_0 = r] + (R_{r+1} - R_r) \\ &= A_r + B_r + (R_{r+1} - R_r) \quad \text{(say)}. \end{aligned}$$

First we compute $\sum_{r=1}^{\infty} \nu^1(r)\, A_r^2$. To this end we insert the event $Y \geqslant pn/2$ in the conditional expectation sign. By Schwarz inequality and the identity $\mathbf{E}_n[c_{\mathrm{zr}}(\eta_x) \mid \eta_0] = \alpha$ the resulting expectation is estimated as follows:

$$\begin{aligned} &(\mathbf{E}_n[(\alpha Y)^{-1}\, c_{\mathrm{zr}}(\eta_x)\, \pi_{x,0}^{\mathrm{zr}} f; Y \geqslant pn/2 \mid \eta_0 = r])^2 \\ &\quad \leqslant \frac{1}{(pn/2)^2\, \alpha} \mathbf{E}_n[c_{\mathrm{zr}}(\eta_x)(\pi_{x,0}^{\mathrm{zr}} f)^2 \mid \eta_0 = r]. \end{aligned}$$

As for the other half we have

$$(\mathbf{E}_n[(\alpha Y)^{-1}\, c_{zr}(\eta_x)\, \pi^{zr}_{x,0} f;\, 1 \leqslant Y < pn/2 \mid \eta_0 = r])^2$$
$$\leqslant \frac{1}{\alpha^2}\, \mathbf{E}_n[c_{zr}(\eta_x)(\pi^{zr}_{x,0} f)^2 \mid \eta_0 = r]\, \mathbf{E}_n[c_{zr}(\eta_x);\, Y < pn/2],$$

so that it is negligible if compared with the first half since the last expectation is exponentially small as $n$ becomes large. Therefore, by employing Schwarz inequality again,

$$\sum_{r=1}^{\infty} \nu^1(r)\, A_r^2 \leqslant \frac{C'}{n} \sum_{x=1}^{n} \mathbf{E}_n[c_{zr}(\eta_x)(\pi^{zr}_{x,0} f)^2].$$

Using Jensen's inequality, the reversibility, Lemma 16 (with $2n$ instead of $n$) together with the property (c) (satisfied by $G_{2n}$) and finally (a) in turn we see that

$$\sum_{x=1}^{n} \mathbf{E}[c_{zr}(\eta_x)(\pi^{zr}_{x,0} f)^2] = \frac{1}{4n^2} \sum_{x=1}^{n} \mathbf{E}[c_{zr}(\eta_0)(\pi^{zr}_{x,0} G_{2n})^2]$$
$$\leqslant C'' n \sup_{b \in \Lambda^*(2n)} \mathscr{D}^b_{p,\rho}(G_{2n}) \leqslant C''' n.$$

(Here $f$ is integrated as a function of $\zeta_n$ and $\eta_0, \eta_1, \ldots, \eta_n$.) Thus we have the bound $\mathbf{E}[\sum_{r=1}^{\infty} \nu^1(r)\, A_r^2] \leqslant C''$.

Next we compute $\sum_{r=1}^{\infty} \nu^1(r)\, B_r^2$. On writing $M - 1 = (\alpha E)^{-1} \times \sum_x (c_{zr}(\eta_x) - \alpha \xi_x)$, it is clear that $\mathbf{E}_n\, |M-1|^2 = O(1/n)$. Hence

$$\sum_{r=1}^{\infty} \nu^1(r)\, B_r^2 \leqslant \frac{C}{n}\, \mathbf{E}_n[f^2 \xi_0].$$

But by virtue of the spectral gap estimate given in Theorem B it follows from the conditions (a) through (c) that

$$\mathbf{E}[f^2] \leqslant C' n. \tag{40}$$

In fact, by employing Jensen's inequality, with the help of the conditions (b) and (c) on $G_{2n}$ Theorem B shows that

$$(4n)^2\, \mathbf{E}[f^2] \leqslant \mathbf{E}[|G_{2n}|^2] \leqslant C n^2 \mathscr{D}^{\Lambda(2n)}_{p,\rho}(G_{2n}).$$

Hence by (a) we conclude (40). It would be clear from this that the sum $\sum_{r=1}^{\infty} \nu^1(r)\, \mathbf{E}[|R_r|^2]$ is negligible. The proof of Lemma 19 is now complete. ∎

**Lemma 20.** $\mathbf{E}[(\mathbf{E}_0[H_{\eta_0} \mid \xi_0 = 1] - H_0)^2] \leqslant C.$

*Proof.* We can proceed as in the preceding lemma with the exclusion operator $\pi^{\mathrm{ex}}_{0,x}$ replacing the zero range operator $\pi^{\mathrm{zr}}_{0,x}$. The argument is much simpler and essentially the same as the corresponding one for the non-gradient exclusion model as treated in ref. 8. ∎

The boundedness of $\mathbf{E}[|s_n^+|^2 c_{01}]$ is now obtained by combining (39) and Lemmas 18 through 20.

## ACKNOWLEDGMENTS

The author is very grateful to an anonymous referee for his valuable and workable suggestions, according to which the tightness result stated in Theorem 1 but missing in the original version has been verified. He is supported in part by the Japan Society for the Promotion of Science No. 15540109.

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